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STATIC AND DYNAMIC MODELLING OF THE SOUTH AFRICAN REAL SECTOR

BY

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P R E F A C E

This thesis provides a detailed investigation of the applicability of various economic models to the South African real sector. It has been written primarily for the statistician, and thus presentation of economic theory has been in a straightforward and, if possible, mathematical form.

Chapter One provides an introduction to the application of econometric methods to economic model building. Chapters Two, Three and Four consider the theory and application of various economic models to the South African case. Chapter Five considers the relevance of money to the real sector.

I would like to thank my supervisor Professor C.G. Troskie for his assistance and support. In addition I would like to Mr. A.M. Hurwitz for his constant encouragement throughout, and for the solving of many of the problems I encountered in my work and for the provision of much relevant information in a field which is so new to South African statistics. Many thanks go to Mrs. M.I. Cousins for organising the typing of the script and for her superb typing of the major portion of it. In addition I would like to thank Mrs. H.B. Light, Mrs. E. Swanepoel and Mrs. B. Tindale for their excellent typing of various chapters.

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CONTENTS

CHAPTER ONE

THE METHODOLOGY OF ECONOMIC MODEL BUILDING

1.1 INTRODUCTION

"Economic theory is necessarily an abstraction from the real world. For one thing the immense complexity of the real economy makes it impossible for us to understand all the interrelationships at once; nor, for that matter, are all these interrelationships of equal importance for the understanding of the particular economic phenomenon under study. The sensible procedure is, therefore, to pick out what appeal to our reason to be the primary factors and relationships relevant to our problem and to focus our attention on these alone. Such a deliberately simplified analytical framework is called an economic model, since it is only a skeletal and rough representation of the actual economy." (Chiang 1974)).

The translation of an economic theory into such an analytical framework necessitates the specification of a number of exact relationships between various economic variables in the form of a set of equations. These equations usually do not take into account all the complex economic interactions, and assume these omitted relationships away in a *ceteris paribus* sense.

To embody all the information that is relevant to these relationships we drop the *ceteris paribus* assumption and incorporate into each equation a random disturbance term. This may be seen as accounting for all the variation in the dependent variable not explained in the "exact" specification. The subsequent specification of some statistical distribution for this random element enables us to analyse the system in a well defined stochastic sense. This allows us to employ statistical methods to estimate the parameters of the model and draw conclusions about their statistical significance according to predetermined confidence levels.

It is sometimes argued, however, that there is no need for a pre-existing body of economic theory. With no postulated cause-effect relationship between variables it is quite possible to draw realistic conclusions about the future behaviour of some variable through its correlation, over some specific time period, with other variables. This would of course avoid the general aim of an empirical economic study which is to gain insight into the mechanism of economic relationships and consequently draw inferences about their long run behaviour.

1.2 SOME BASIC TERMINOLOGY

It seems appropriate at this stage to familiarise the unacquainted reader with some terminology that has specific relevance in the field of economic modelling.

Given any economic model we may split the variables into those that are endogenous and exogenous to the system. Endogenous or jointly dependent variables are those whose values are determined within the bounds of the system. Exogenous variables, in which category we may include lagged endogenous variables, are those which are completely predetermined and not affected by any (economic) relationships within the model. Both variable types may be given a direct causal interpretation in the model, but it is only the endogenous variables that are in turn seen to be affected.

The equations of the system may be divided into identities and behavioural relationships. An identity expresses a tautology, that is, something which is true by definition. For example, total profit is defined as the excess of total revenue over total cost; we can therefore write the identity

$$\text{Profit} \equiv^{*} \text{Revenue} - \text{Cost}$$

A behavioural equation expresses the direction and manner of causal influence that a set of variables exerts on some endogenous variable. For example

$$C = \beta_0 + \beta_1 DY + \beta_2 W; \quad \beta_1, \beta_2 > 0$$

where C is the consumption of ice cream

DY is disposable income

W represents the weather

asserts that changes in disposable income or the weather will

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*For such an equation the identical equality sign $\equiv$ is usually employed in place of the regular equals sign $=$.

cause the consumption of ice cream to be affected.

Finally, it is thought relevant in this section to give a brief discussion of the three broad categories of economic analysis: static; comparative static and dynamic analysis. Static analysis involves the calculation of the equilibrium levels of a set of economic variables that form some interdependent system.

eg. Consider the simple market demand model.

$$Q_s = aP \quad ; \qquad a > 0$$
$$Q_d = b - cP \quad ; \qquad b,c > 0 \qquad (1.2.1)$$
$$Q_s = Q_d \quad \text{(Equilibrium condition)}$$

where P is the Price Level

Q_s, Q_d are the quantity supplied and demanded respectively.

This yields the solutions

$$\bar{P} = \frac{b}{a+c}$$

$$\bar{Q} = \frac{ab}{a+c}$$

where $\bar{P}$ and $\bar{Q}$ represent the equilibrium values of P and Q.

These equations yield limited information about the system. They are only true for the one point in time that the equations describing Q_s and Q_d hold. Of much greater value would be to conduct a comparative static analysis in which one examines the consequences on equilibrium of a change in the parameters a,b and c.

For example, computation of the partial derivatives

$$\frac{\partial \bar{P}}{\partial a} = \frac{-b}{(a+c)^2} < 0$$

$$\frac{\partial \bar{Q}}{\partial a} = \frac{bc}{(a+c)^2} > 0$$

indicate that an increase in the parameter a yields a decrease in equilibrium price but an increase in the quantity cleared in the market.

Comparative static analysis does, however, abstract from time and hence provides no information regarding the movement towards equilibrium through continuous time or even whether equilibrium is in fact attained. In the model (1.2.1) we could incorporate the equation

$$\frac{dP}{dt} = \alpha(Q_d - Q_s)$$

where α represents a constant speed of adjustment coefficient ($\alpha > 0$). That is, that the rate of change of price is always directly proportional to the excess demand $Q_d - Q_s$.

Solution of this system yields

$$P(t) = [P(0) - \bar{P}]e^{-\alpha(a+c)t} + \bar{P}$$

where $P(0)$ is the initial price and $\bar{P}$ is the market clearing price as before.

This solution infers that for any $P(0)$ equilibrium will be attainable for large t and that in fact the system is dynamically stable because it will converge to the market clearing price $\bar{P}$.

1.3 THE PROBLEMS OF SPECIFICATION

1.3.1 SPECIFICATION 'A PRIORI'

The presupposition of knowledge pertaining to the establishment of an economic model enables the econometrician to formulate *a priori*, theoretical expectations about the sign and the size of parameters in the model. For example we might write

$$C = \beta_0 + \beta_1 DY + \beta_2 P + u$$

where C is consumption expenditure on a durable good
DY is disposable income
P is price
u is a random error term

If we are dealing with a normal, (in the economic sense), good we would postulate, *a priori* that $0 < \beta_1 < 1$ and $\beta_2 < 0$. That is, that expenditure on the good could at most comprise our entire disposable income, and that price increases would result in decreased consumption through substitution and real income effects. However, if we were to study the consumption expenditure of some ultra-inferior good (in the economic sense) we would postulate, *a priori* that $\beta_1 < 0$ and $\beta_2 > 0$. The explanation being that increases in disposable income would result in other goods being substituted for the good in question and that for increases in price, real income effects would swamp substitution effects and result in increased consumption.

1.3.2 SINGLE OR MULTI-EQUATION MODEL

Economic theory is not clear in dictating whether phenomena should be viewed in terms of single equation or simultaneous multi-equation models. The intricacy of real phenomena might lead one to immediately frame one's argument in terms of large simultaneous systems. From a theoretical angle, simultaneity often leads to problems of interpretation of causal dependencies, but this does not seem to have detracted from their general acceptance in economics.

Estimation of simultaneous systems, however, involves more sophisticated statistical techniques and allowsmuch less flexibility of specification than single equation models. Consequently a large part of econometric research has been based on single equation models and estimation by single equation techniques.

Of special relevance to simultaneous systems is the imposition on any particular equation of *a priori* restrictions in the form of included or excluded variables. The exclusion of a variable is the imposition, *a priori* of a zero coefficient on that variable. Similarly the inclusion of a variable is the imposition, *a priori* of a non-zero coefficient on the particular variable.

We might remark that possibly the weakest point of econometric application is in the specification of the basic economic model. Economic theory, especially that pertaining to stabilization policy, is often shrouded in ambiguity and ill-defined statements. Until there is a complete formalization of ideas behind economic model building, one cannot expect any reconciliation of the conflicting theories with which economic thinking is traditionally fraught, and hence one may only speculate on which policy actions might be most desirable.

1.4 THE RELEVANCE OF CAUSALITY TO ECONOMIC MODELLING

A law of nature is an empirical regularity which we discover by careful observation of certain phenomena. A law might, however, give no insight into the cause and effect relations behind the behaviour observed and the regularities detected. It is the explanation of these relations that is of fundamental importance in economic science, and vital if we wish to predict or control economic phenomena under changing circumstances where parameters vary but the underlying cause-effect relationships do not.

A cause-effect relation in a system can be conceived of as a connection between 2 parts, A and B of the system. The connection is such that an occurrence or change in A is followed by an occurrence or change in B. There is thus a fundamental asymmetry in a cause-effect relation as distinct from a simple correlation. The unidirectional aspect of the causal relation is a vital concept in basic economic relationships; for example the fundamental issue of the Monetarism versus Keynesian debate centres on the direction of causality between Money and Income.

Consider a system of p interdependent equations of the form

$$Ay + Bz = u$$

where y is a p vector of endogenous variable

z is a k vector of exogenous variable

A is a $p \times p$ vector of endogenous coefficients

B is a $p \times k$ vector of exogenous coefficients

u is a random disturbance vector

This system cannot generally be interpreted as a set of unilateral causal relations in which each equation describes the response of one variable provided by the other variables, unless A is an upper right triangular matrix. In this case the system may be given a causal interpretation, the interpretation being that y_p is causally determined by the exogenous variables z_i; y_{p-1} causally determined by the (already determined) y_p and the z_i, and in general y_i is causally determined by $y_{i+1},\ldots,y_p$ and the z_i.

The acceptance of static general equilibrium models of the Walrasian and Keynesian type which are simultaneous systems with A non-triangular have, however, provided problems of causal interpretability. These systems of "non recursive" equations raise problems of causal circles between the endogenous variables. Strotz and Wold (1960) have argued, however, that these non-recursive systems can be interpreted in a recursive sense by virtue of the fact that they are a reduced form of an underlying recursive system.

Consider the model

$$C_t = \beta_0 + \beta_1 Y_t$$
$$Y_t = C_t + I_t \tag{1.4.1}$$

where C_t is consumption at time t

Y_t is income at time t

I_t is investment at time t

A relationship of the form $C_t = f(Y_t)$ tends to imply an instantaneous relation between Y_t and C_t. More realistically we could assume a slight time delay for the cause to produce the effect. We might write $C_t = f(Y_{t-\theta})$ expressing a delay of θ time units from cause to effect. The above system could then be given the causal interpretation

$$Y_{t-\theta} \rightarrow C_t \rightarrow Y_t$$

If however θ is small we could write $C_t = f(Y_t)$ and understand that such a causal relation implies a slight delay between cause and effect. This infers that an economic model consisting of interdependent equations of the form (1.4.1) with all endogenous variables dated at time t is not really simultaneously interdependent. It is merely that the small time lags between cause and effect throughout the system have been obscured in the switch from continuous to discrete time.

With this in mind we can give an interpretation of bi-causality between C and Y in (1.4.1) by splitting the system into 2 parts. In the behavioural part of the model Y exerts a direct causal influence on C while C exerts a direct causal influence on Y in the identity part of the system. Of additional interest in such a system would be the rates at which the direct causal variables change the effect variable. Referring to linear systems of the type (1.4.1) the partial derivative $\frac{\partial Y}{\partial C}$ would give the change in the effect variable Y, per unit change of the direct causal variable (C). In order to remove the effects of the units of measure used for C and Y we may calculate point elasticities $\frac{\partial Y}{\partial C} \cdot \frac{C}{Y}$ which measure the percentage change in the effect variable stemming from a 1 unit percentage change in the causal variable.

Referring back to our system (1.4.1) we might note that if a lag in causal dependency between endogenous variables is smaller than the intervals between the observations of the variables, a simultaneous equation model in which each equation may involve causal endogenous variables will often result in a less serious error of specification than if each equation is made a function of exogenous variables. This leads on to a critical problem in practical econometric work. This is that macro-economic data comprises series of observations made at much longer intervals than the intervals between the decisions they reflect. Macro-economic theory is only the theory of aggregate micro-economic decision making; true causation is generally between micro-economic variables; the macro cause-effect relationships simply portray an aggregation of micro-cause-effect relations.

Our goal is to construct models in which the variables are allowed to change at realistically short intervals, and to make inferences about the parameters of these models from observations at longer time intervals.

Since the variables that occur in most econometric models are the result of large numbers of micro-economic decisions taken by different individuals at different points of time, they may for practical purposes be regarded as continuous functions of time. The models developed in Chapters 3 and 4 of this thesis provide the means of making statistical inferences from discrete data about the parameters of models whose variables are continuous functions of time.

The models take the form

$$Dy(t) = F[y(t), z(t), \theta] + u(t); \quad D = \frac{d}{dt}$$

where $y(t)$ and $z(t)$ are random functions of time but observable at discrete intervals, θ is a vector of parameters and $u(t)$ is a vector of white noise disturbance. In the words of Wymer (1976)

> "This model is recursive in the sense that the derivative of each endogenous variable is determined by a function of the levels of all variables and thus may be given a causal interpretation."

If we can obtain efficient estimates of θ we can obtain a prediction formula for y in terms of continuous t, which could be used for forecasting the values of y at any t even though y is observable only at discrete intervals of time.

1.5 SUMMARY AND CONCLUSIONS

It is seen that the necessary discretization of continuous time in linear models leads to an unrealistic representation of an economic system. The use of a simultaneous system with endogenous variables appearing as explanatory variables is more realistic but leads to problems of causal interpretability. Continuous time models provide possibly the most acceptable solution particularly from an economic-theoretic viewpoint; computation is however burdensome, and the models prove extremely sensitive to even minor specification changes. In contrast the statistical analysis of zero order linear systems is straightforward and provides for easy manipulation of equations.

For these reasons it seemed appropriate to examine the applicability of both linear and differential equation system models to the South African economy. In Chapter 2 a linear model is presented and various problems associated with the equations, notably those of multicollinearity and autocorrelation are discussed in detail. Chapter 3 provides a full theoretical exposition of the theory of continuous time differential equation models. In Chapter 4 two different equation models are applied to the South African case, the first is linear, the second is in log-linear form. Finally in Chapter 5 the current Keynesian-Monetarist debate is given a mathematical exposition and a discussion of the applicability of these competing theories to the South African real sector is presented.

CHAPTER TWO

A LINEAR FORECASTING MODEL FOR THE SOUTH AFRICAN REAL SECTOR

2.1 INTRODUCTION

In the South African context, because of the lack of research in the field, it was considered to be of value to develop a linear forecasting model of the real sector. It was assumed that in conjunction with the linear model of the monetary sector developed by Hurwitz and Kantor (1977), a model with reasonable predictive power could be established for the combined monetary and real sectors of the economy. The methodological approach is straightforward, relying on the existence of strong correlation between trends in the flow and stock variables that relate to the real sector. It is not intended that this model provide us with a theoretical description of the economy. The partial equilibrium system developed later in this text will go some way to filling that gap. The more complex a system is, the more difficult it is to manipulate it, and the more sensitive it is to minor changes in specification. With a linear system, estimated with ordinary least squares one can modify and manipulate and still have a very accurate idea of the contribution in explanatory power of each variable and their statistical significance etc.

The work on this forecasting model was stimulated by research done by de Wet and Dreyer (1977). They attempted, however, to incorporate more economic theory into their behavioural equations which required that in many, they had a large number of correlated independent variables. However, in their analysis they made no adjustment for multicollinearity. This writer concentrated on using a small number of explanatory variables and achieved a very good explanation; where it was necessary to use a number of explanatory variables care was taken to adjust for multicollinearity. The analysis of multicollinearity has attracted considerable interest (Troskie (1971), Wong Fung (1977), and Coutsourides (1977)) at the University of Cape Town. Its practical applicability has, however, not been so fully examined and it was felt that such a linear system could in some way provide a testing ground for such an applied study. A computer programme for the solution of the problem of multicollinearity (the Ridge solution and the modified least square-characteristic root solution) was written by the author - see Appendix C, and was used in the analysis below. In addition use was made of the packages ECON and AUTO available on the UNIVAC 1106 at the University of Cape Town for Ordinary Least Squares and autocorrelation analysis.

2.2 SPECIFICATION OF THE MODEL

The model can be represented in the following summary form where the f_i; $i = 1,\dots,9$ denote linear functions. It consists of nine behavioural equations and five identities - the

u_i; $i = 1,\dots,9$ represent stochastic error terms.

1. $C = f_1(DY, S4) + u_1$
2. $LNA = f_2(GNA_{-1}, KNA_{-1}, S1, S4) + u_2$
3. $I = f_3(Y_{-1}, i_{LT}) + u_3$
4. $Imp = f_4(Imp_{-1}, Gde) + u_4$
5. $Exp = f_5\left(Y_{-1}, \left(\frac{P_E}{P}\right)_{-1}\right) + u_5$
6. $P_eTAX = f_6(W, S_2, S_4) + u_6$
7. $W = f_7(Y_{-1}) + u_7$
8. $S = f_8(Exp, Y_{-1}, S_4) + u_8$
9. $\Delta P = f_9(\Delta M_S, \Delta P_{Imp}) + u_9$
10. $DY \equiv W + R - P_eTAX$
11. $\Delta S \equiv S - S_{-1}$
12. $P \equiv \Delta P + P_{-1}$
13. $Y \equiv C + I + G + X - Imp + \Delta S$
14. $Gde \equiv C + I + G + \Delta S$

ENDOGENOUS VARIABLES

LNA = employment in non-agricultural sector

C = private consumption expenditure

I = Gross domestic fixed investment

Imp = imports of goods and non-factor services

Exp = exports of goods and non-factor services

P_eTAX = personal tax

W = wages

S = inventory level

ΔS = change in level of inventories

P = price level

ΔP = change in prices

DY = disposable income

Y = gross national income

Gde = gross domestic expenditure

EXOGENOUS VARIABLES

KNA_{-1} = fixed capital stock in non-agricultural sector lagged 1

GNA_{-1} = gross domestic product in non-agricultural sector lagged 1

i_{LT} = real long term interest rate

$(P_E/P)_{-1}$ = ratio of export price to domestic price level lagged 1

ΔM_s = change in money supply

ΔP_{Imp} = change in import price

$S1$ = dummy first quarter seasonal

$S4$ = dummy fourth quarter seasonal

R = transfers to the private sector and income from property by households

Any variable with the subscript $_{(-1)}$ denotes that the variable is lagged by 1 time period - in the case of endogenous variables the variable will be pre-determined.

The specification above was that considered to yield the most accurate representation in each case (on the basis of R^2, t statistics etc.). However, a number of variations were tested and the estimation results are given below.

2.3 THE ECONOMIC JUSTIFICATION

1. The theory of aggregate consumption behaviour is simply the aggregation of the theory of the individual's comsumption behaviour. For the individual, and hence in the aggregate we postulate

$$C = \alpha DY + \beta; \qquad \alpha > o \qquad \beta > o$$

The positive value of β (obtained below in the aggregate function) infers that

(i) at low rates of disposable income the individual will dissave (or consume out of past savings).

(ii) $\frac{C}{DY} = \alpha + \frac{\beta}{DY}$

decreases as disposable income increases approaching the level α. This means that the proportion of disposable income spent on consumption decreases as disposable income increases - one would expect, for example, rich people's consumption expenditure to be a smaller percentage of their disposable income than in the case of poorer people.

(iii) The disposable income elasticity of consumption

$$= \frac{\alpha C}{\alpha DY} \cdot \frac{DY}{C}$$

$$= \alpha(\frac{1}{\alpha} - \frac{\beta}{C\alpha})$$

$$= \frac{\alpha DY}{\alpha DY+\beta}$$

which approaches 1 for large DY.

Thus, for large disposable income a 1% increase in disposable income would cause a 1% increase in consumption. One might expect that this elasticity would depend on the aggregated consumption - utility map and not be constant but rather some function of expectations (especially of future income) as in the Fredmanite theory (M. Friedman : A theory of the Consumption Function (1957)). The conventional utility Map would infer, in particular that the elasticity is zero past some point C_0 (see Figure 2.1)

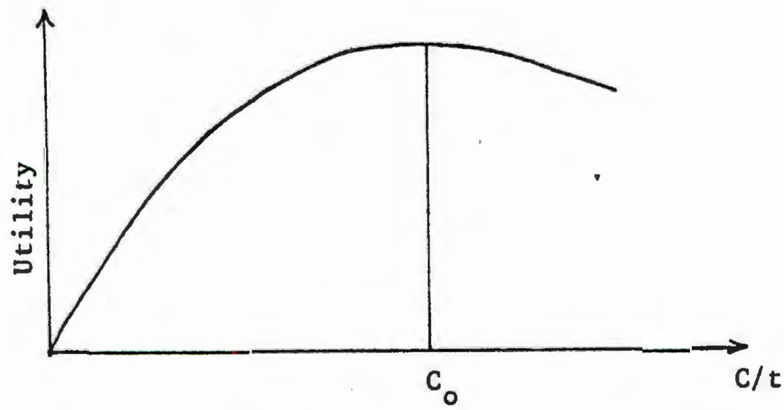


Figure 2.1 Graph of Utility versus aggregated consumption

The high explanation, and strong t-statistic (see below) for DY point to the fact, however, that this representation was satisfactory for the South African case.

Initially a short term rate of interest was included in the equation on the assumption that, as a considerable amount of consumption expenditure is made on credit, the cost of credit should be negatively related to consumption flow. This parameter was found to be non-significant and of the incorrect sign (see below). In the case of consumption expenditure expectations of high and possibly increasing inflation rates might prompt people to make purchases irrespective of credit costs, which would, in any case be significantly lower than the inflation rates of most durable goods. A fourth quarter dummy variable was included to take into account the above-average consumption expenditure in the Christmas period.

2. We hypothesise that labour in the non-agricultural sector (total labour force data unavailable) is a function of lagged captial stock in the non-agricultural sector plus lagged gross domestic product in non-agricultural sector, plus seasonal dummies for the first and fourth quarters. The seasonal dummies compensate for the lower labour requirements during the Christmas period extending into mid-January as manufacturing and building cut down on their activities.

A function that relates income to the factors of production (Land usually omitted) is termed a "production function". The one postulated here takes on a linear form which does not correspond to conventional economic theory. In order to avoid linearization problems (having to convert non-linear forms to linear forms by Taylor series expansions about the means) it was decided to use a linear function. Production functions in estimation work usually take on one of the two following forms (Chiang (1974)).

$$Y = AK^{\alpha}L^{\beta}$$

or alternatively $L = A^{-\frac{1}{\beta}} Y^{\frac{1}{\beta}} K^{-\frac{\alpha}{\beta}}$

where Y = income

K = capital stock

L = labour

A, α, β constants > 0.

This production function is homogeneous of degree $\alpha + \beta$. Note that f is "homogeneous of degree r" if for each constant k:

$$f(kx_1, kx_2, \ldots, kx_n) = k^r f(x_1, x_2, \ldots, x_n)$$

i.e. a k^{th} fractional increase in K and L will result in a $k^{\alpha+\beta}$ increase in Y.

In the special case $\alpha + \beta = 1$ the production function is known as a Cobb-Douglas Production function and is linearly homogeneous. In addition the isoquants of this function (the two dimensional plots of a particular constant Y against varying K and L) are negatively sloped throughout and strictly convex for positive K and L (i.e. $\frac{dK}{dL} < 0$ and $\frac{d^2K}{dL^2} > 0$), (see Figure 2.2).

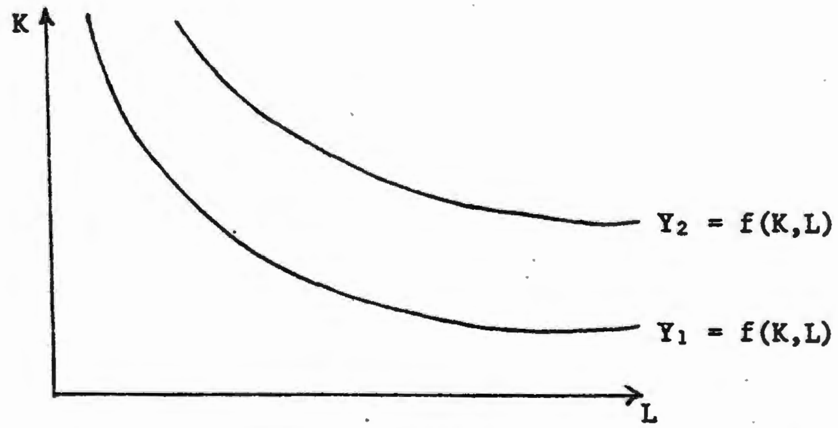


Figure 2.2 Isoquant plots for Cobb-Douglas production function

The second form are those known as "constant elasticity of substitution" (CES) production functions. Written as

$$Y = A[\delta K^{-p} + (1-\delta)L^{-p}]^{-\frac{1}{p}}$$

$$A > 0 \quad 0 < \delta < 1 \quad p > -1$$

These functions are homogeneous of degree 1 and as with the above have $\frac{dK}{dL} < 0$ and $\frac{d^2K}{dL^2} > 0$.

The constant elasticity of substitution follows from the fact that

$$\ell n\left(\frac{\bar{K}}{\bar{L}}\right) = \ell n\ c + \frac{1}{1+p}\ \ell n\left(\frac{P_L}{P_K}\right) \quad \text{(Chiang (1974))}$$

where c is a constant, P_L, P_K are the prices of L and K respectively, and $\frac{\bar{K}}{\bar{L}}$ is the optimal factor ratio, i.e. where

$$\frac{\text{marginal productivity of capital}}{\text{marginal productivity of labour}} = \frac{\text{price of capital}}{\text{price of labour}} .$$

The elasticity of substitution

$$= \frac{d\ \ell n\left(\frac{\bar{K}}{\bar{L}}\right)}{d\ \ell n\left(\frac{P_L}{P_K}\right)} = 1 + p \text{ (constant)}$$

As mentioned before the problems (see note on elasticity - Chapter 4) of linearizing a non-linear (linear in terms of logs) form in the case of simultaneous estimation of a linear system are considerable, and as explanation of labour in a linear system was so good (99%) it was decided to use the (theoretically) less sophisticated linear form.

It was of some interest that in the model of de Wet and

Dreyer (1977) great emphasis was given to the fact that, a priori, a negative sign for capital stock was desirable. In a linear system it seems unreasonable to assume that rising K be accompanied by falling L - one would in fact expect their movement to be closely correlated. In the industrial sector, in South Africa for example, very few labour constraints exist and the implementation of new capital equipment would be accompanied by the employment of additional labour up to the point that was desirable. It would be unreasonable to expect that in the short run when income was reasonably constant a linear system could pick up substitution effects, for example, a shift from capital to labour because of falling wages and rising interest rates.

3. Our eventual specification made investment a linear function of lagged income and the real long term rate of interest. Economic theory would in general dictate that capital stock and not investment be closely correlated to income. Given any stock of capital, and marginal rates of substitution between factors of production that are nearly constant, the flow of output should be closely related to the stock producing that output. From this it would follow that change in K, i.e. investment, is a function of change in Y, which is the accelerator theory of investment. This theory was tested (see below) but the regression explanations obtained were disappointing. Investment is, according to classical theory, a declining function of the interest rate; the interest rate is the cost of finance or the opportunity cost of holding funds, and one

might expect a high cost of finance to result in declining investment. However, since investment decisions in South Africa are so closely related to expectations of political events, that is riskiness of the venture, and as high rates of return are required to offset the risk of capital loss, one could expect the Investment schedule to be relatively interest inelastic. In fact, one could almost say that Investment was a function of some set of political dummy variables alone. In addition, it seemed realistic, if the interest rate was to be included, to consider the "real" rather than the nominal interest rate in order to obtain a truer representation of the cost function involved. An attempt was also made to estimate an expectations model of the form

$$I_t = \gamma(K_t^* - K_{t-1})$$

where

γ is a constant

K_t^* is desired capital stock at time t

K_{t-1} is actual capital stock at time $t-1$

I_t is investment flow at time t.

Investment is assumed to be some constant proportion of the difference between desired and actual capital stock. Desired capital stock might be taken to be a linear function of income and the rate of interest (Koutsoyiannis (1973)).

Use of this specification resulted in unsatisfactory parameter estimates, however, (see below). When used in a simultaneous context a model very similar to the above was estimated satisfactorily using Full Information Maximum Likelihood (Chapter 4).

4. Imports are assumed to be linear function of Imports lagged 1 and gross domestic expenditure. This representation is equivalent to the assumption that imports are a linear function of current expenditure and an infinitely declining series of lags in expenditure (Koyck (1954)).

Writing

$$Imp_t = \beta_0 + \sum_{i=0}^{\infty} \lambda^i \beta_1 Gde_{t-i} + u_t \qquad 0 < \lambda < 1$$

Lagging one period and multiplying by λ

$$\lambda \, Imp_{t-1} = \beta_0 \lambda + \sum_{i=0}^{\infty} \lambda^{i+1} \beta_1 \, Gde_{t-i-1} + \lambda u_{t-1}$$

Subtracting we obtain

$$Imp_t = \beta_0 (1-\lambda) + \lambda \, Imp_{t-1} + \beta_1 \, Gde_t + v_t$$

where $\quad v_t = u_t - \lambda u_{t-1}$

Note: Even if the original specification is correct, that is, u_t random, v_t will be autocorrelated (Appendix A).
One of the features of the South African economy has been close correlation through the business cycle between expenditure and imports. The imposition of the import surcharge recently could, however, easily disturb this relationship.

5. The bulk of South Africa's exports are in the form of raw materials and semi-processed raw materials. The prices of these commodities are controlled on world markets and are affected in the obvious way by the forces of demand and supply. The supply of these commodities by South African producers will depend in the first instance on the ratio P_E/P (price of

South African exports over local prices), the higher this ratio the greater the amount supplied to foreign markets in preference to local markets. In addition one could expect exports to depend on local economic activity or income, the higher national income the higher the level of production (supply) of raw materials and manufactured goods and hence the export potential.

6. Tax is taken to be a linear function of income and seasonal variation. Institutional factors would dictate that the constant term be negative, that is, of the form

$$T = \gamma Y - B \qquad \gamma, B > 0$$

$$\frac{T}{Y} = \gamma - \frac{B}{Y}$$

Therefore as Y increases the ratio $\frac{T}{Y}$ gets larger - this is a reflection of progressive tax rates. Seasonal variation is a result of institutional factors determining time of payment.

7. Wages are taken to be a linear function of lagged income. Specifications involving current and lagged income were tried (see below) but did not give more satisfactory explanation. This relationship is particularly stable in the South African case because of the absence of black trade unions in all major industrial and commercial enterprises and their consequent lack of power in wage determination.

8. Inventories are taken to be a positive linear function of income and seasonal variation and a negative linear function of exports. A number of agricultural products are stored as in-

ventories because of price fluctuations and export of these commodities takes place to a large extent out of these inventories.

Inventory level would normally be assumed to have some negative correlation to the cost of finance, a high short term interest rate being associated with a reluctance to build up inventories and vice versa. Firms, however, have to weigh the finance costs of stock against the advantages of holding stock. Having a reserve of finished products allows them to meet unexpected demands; with a stock of materials, they are in a position to step up output at short notice, but these facilities will become relatively less attractive if their costs rise and more attractive if their costs fall. It appears though that even if stock demand is elastic it is so with respect to total holding cost, that is, warehouses,personnel etc. whose expenses will be considerably greater than finance charges, and thus stock demand will be inelastic with respect to them.

9. Predicting changes in levels of stock variables is a much more satisfactory test of economic theory than the prediction of levels because we abstract from trend correlations which will in general exist. Changes in prices were found to be satisfactorily explained by changes in Import Prices and Money. An additional specification involved explanation via inventory changes, that is, high prices associated with low inventory levels (negative changes in inventories) and vice versa. A monetarist would argue that this is a duplication of theory,

however, because money is a strong reflection of demand and thus inventory level, so that a specification in terms of both variables would be redundant. Monetarist or Neo-Keynesian it is completely accepted that changes in money effect demand and then prices. South Africa imports considerably from America, Western Europe and Japan, and hence will import inflation directly in the case of consumer items and indirectly in the case of capital stock.

2.4 ESTIMATION

As mentioned above a number of alternative specifications were considered before the final form was decided upon. All regressions were carried out on the UNIVAC 1106 using the Ordinary Least Squares (O.L.S.) estimation package ECON. All estimations were made using quarterly data at constant 1970 prices over the period 1960-1974 (see Appendix B) except in the case of one of the investment equations which used Full Information Maximum Likelihood - programmes and estimation procedure discussed in detail in Chapter 4.

All these results are presented below with the relevant ordinary least square summary statistics.

The summary statistics, which are given below are discussed in detail in Appendix A.

(i) t-statistics, which are given below each coefficient. Each t-statistic has n-k degrees of freedom where k is the number of explanatory variables plus one, and n the number of observations.

(ii) R^2 - multiple coefficient of determination adjusted for degrees of freedom.

(iii) D.W. - the Durbin-Watson statistic.

(iv) d.o.f. - the degrees of freedom.

(v) S.E. - standard error of the estimate.

$$**C = 10{,}1426 + 0{,}8517.DY + 87{,}1676.S4$$
$$\quad (0{,}223) \quad (35{,}116) \quad (3{,}538)$$

R^2 = 0,9572 D.W. = 2,2190 S.E. = 81,8668

d.o.f. = 55

$$C = -5{,}1176 + 0{,}8379.DY + 10{,}7811.i_{ST}$$
$$\quad (-0{,}105) \quad (29{,}227) \quad (0{,}904)$$
$$+ 79{,}055.S4$$
$$(3{,}091)$$

R^2 = 0,9574 S.E. = 83,0279 D.W. = 1,7928

d.o.f. = 55

$$**L = 15737{,}9970 + 0{,}1241.KNA_{-1} + 6{,}9619.GNA_{-1} - 616{,}9029.S1$$
$$\quad (61{,}773) \quad (2{,}114) \quad (11{,}089) \quad (-3{,}640)$$
$$-249{,}8652.S4$$
$$(-1.596)$$

R^2 = 0,9903 S.E. = 487,9245 D.W. = 0,4486

d.o.f. = 53

$$**I = -247{,}6727 + 0{,}3172.Y_{-1} - 1{,}7341.i_{LT}$$
$$(-8{,}605) \quad (33{,}188) \quad (-1{,}135)$$

R^2 = 0,9660 S.E. = 42,3130 D.W. = 1,6927

d.o.f. = 55

$$I = -252{,}5301 + 0{,}2147.Y_{-1} + 0{,}1256.Y_{-2}$$
$$(-8{,}984) \quad (3{,}504) \quad (2{,}024)$$
$$-2{,}2187\ i_{LT}$$
$$(-1.474)$$

R^2 = 0,9678 S.E. = 41,1703 D.W. = 1,4538

d.o.f. = 54

$$I = 604{,}2284 + 0{,}2558.\Delta Y - 27{,}738.i_{LT}$$
$$(22{,}669) \quad (0{,}915) \quad (-5{,}698)$$

R^2 = 0,3449 S.E. = 187,1446 D.W. = 0,3577

d.o.f. = 55

$$K = -944{,}0238 + 1{,}6458.Y_{-1} + 24{,}052.i_{LT}$$
$$(-7{,}966) \quad (36{,}067) \quad (3{,}883)$$

R^2 = 0,970 S.E. = 186,738 D.W. = 1,06

d.o.f. = 55

(ESTIMATION BY METHOD OF FULL INFORMATION MAXIMUM LIKELIHOOD)

$$I = 19{,}4994 + 0{,}0126\ (26{,}8150 \cdot Y + 160{,}5005.i_{LT} - K_{-1})$$
$$(0{,}28) \quad (1{,}18) \quad (2{,}41) \quad (0{,}11)$$

$$**Imp = -26{,}6943 + 0{,}8257.Imp_{-1} + 0{,}05704.Gde$$
$$(-0{,}904) \quad (11{,}315) \quad (1{,}408)$$

$R^2 = 0,9546$ S.E. = 42,5560 D.W. = 1,442

d.o.f. = 55

$$Imp = -19,0718 + 0,1553.Gde + 0,1046.Gde_{-1}$$
$$(0,510) \quad (1,824) \quad (1,222)$$

$R^2 = 0,8528$ S.E. = 76,6008 D.W. = 0,3945

d.o.f. = 55

$$**Exp = -525,0298 + 0,2119.Y_{-1} + 592,279.(P_E/P)_{-1}$$
$$(-7,065) \quad (22,183) \quad (9,399)$$

$R^2 = 0,9106$ S.E. = 49,3001 D.W. = 1,5817

d.o.f. = 55

$$**P_eTAX = -13,3510 + 0,1179.W - 60,5012.S2$$
$$(-0,794) \quad (10,576) \quad (-5,773)$$
$$-46,8852.S4$$
$$(-4,573)$$

$R^2 = 0,7215$ S.E. = 32,2629 D.W. = 2,6693

d.o.f. = 54

$$**W = -53,1033 + 0,5558.Y_{-1}$$
$$(57,470) \quad (-2,006)$$

$R^2 = 0,9830$ S.E. = 49,9152 D.W. = 0,830

d.o.f. = 56

$$W = -50,8974 + 0,3400.Y_{-1} + 0,21794.Y_{-2}$$
$$(-2,070) \quad (4,933) \quad (3,158)$$

$R^2 = 0,9854$ S.E. = 46,34 D.W. = 0,3641

d.o.f. = 55

$$S = -605,9512 + 1,7338.Y - 0,9073.Exp$$
$$(-5,50) \quad (21,966) \quad (12,773)$$
$$-85,667.S4$$
$$(-1.477)$$

$R^2 = 0,9686$ S.E. = 189,558 D.W. = 1,3921

d.o.f. = 54

$$**S = -566,55173 + 1,73096.Y_{-1} - 0,87321.Exp$$
$$(-4,698) \quad (20,420) \quad (-2,480)$$
$$-26,49615.S4$$
$$(-0,950)$$

$R^2 = 0,9643$ S.E. = 202,3114 D.W. = 1,7831

d.o.f. = 54

$$\Delta P = 0,4557 - 0,001536.\Delta S + 0,3588.\Delta P_{Imp}$$
$$(3,842) \quad (-1,727) \quad (10,822)$$
$$+ 0,002485.\Delta M_s$$
$$(3,527)$$

$R^2 = 0,8619$ S.E. = 0,6536 D.W. = 2,1937

d.o.f. = 54

$$**\Delta P = 0,1876 + 0,5415.\Delta P_{Imp} + 0,00356.\Delta M_s$$
$$(1,436) \quad (13,869) \quad (4,295)$$

R^2 = 0,8394   S.E. = 0,7746   D.W. = 2,2574

d.o.f. = 55

The estimations marked ** were taken as the most satisfactory O.L.S. equations and used for forecasting purposes.

The one sided 95% point of the t-statistic with 56 degrees of freedom is 1,673 (or -1,673 for a left sided test).

Considering the equations ** (see above) it is seen that the constant in the C equation is non-significant. In addition S4 in the labour equation and inventory equation i_{LT} in the investment equation and Gde in the Import equation are marginally non-significant. However, economic theory dictates that they should be included on the grounds that exogenous factors (especially political in the case of the Investment equation) specific to the sampling period could distort the strength of influence of the explanatory variables. In addition it is known that the presence of multicollinearity results in small t-statistics (through parameter estimates having large variances) and could lead to excluding variables that have significant explanatory power.

Consideration of the Durbin-Watson statistic (see Appendix A) yields the following significant points for n = 55 at the 95% level for positive autocorrelation.

K = 1		K = 2		K = 3		K = 4	
dL	dU	dL	dU	dL	dU	dL	dU
1,53	1,60	1,49	1,64	1,45	1,68	1,41	1,72

Referring to the equations estimated it is seen that the Labour, Import and Wage equations exhibit significant positive first order autocorrelation. Note, however, (Appendix A) that in the case of the import equation the Durbin-Watson statistic is biased. Durbin has in fact developed a test (when $n \geq 50$) for determining the presence of autocorrelation when a lagged dependent variable is used as an explanatory variable. Unfortunately one of the conditions of the test was not satisfied, namely that $n\,V(\beta) \leq 1$ (where $V(\beta)$ is the variance of the coefficient of the lagged dependent variable and n the number of observations, (see Appendix A for details).) In a Monte Carlo study of a model with a lagged dependent variable, however, Houthakker and Taylor (1970) concluded that a reasonable criterion is to accept the null hypothesis that no autocorrelation exists if the Durbin-Watson statistic lies outside the limits put forward in the conventional D.W. tables.

An analysis to correct for first order correlation is carried out below using the Cochrane Orcutt method.

2.5 ALMON LAG ANALYSIS OF THE CONSUMPTION FUNCTION

A more reasonable assumption than making consumption a function of disposable income, would be to make it a function of current income plus lagged values of income, that is, consumption is not only dependent on current income but also on income received in previous periods.

In general then

$$C = f(DY_t, DY_{t-1}, \ldots, DY_{t-n}) \quad \text{for some } n.$$

The method of Almon Lag Analysis was undertaken which reduces multicollinearity and preserves d.o.f. by making use of polynomial approximations (of degree s say) to the coefficients of the lagged variables where $s \leq n$ (see Appendix A).

Three different specifications were tried using an n of 3,5 and 7, with a polynomial of degree 3. The coefficients of the respective variables are given in Table 2.1.

The 5% one sided t-statistic critical point with 48 d.o.f. is 1,679 (or -1,679 for the test of a negative coefficient). The SUM column indicates the explanatory power of the entire set of lags involved in the particular regression.

It is clear from Table 2.1 that disposable income has a significant effect on income in the current period and after a lag of one and two periods. After that, however, it seems that the regressions are picking up extraneous correlation and yielding negative signed or positive signed coefficients in an inconsistent pattern (often with weak t-values).

TABLE 2.1

n	C	C_{-1}	C_{-2}	C_{-3}	C_{-4}	C_{-5}	C_{-6}	C_{-7}	Const.	SUM	R^2	SE	df
3	0,4900	0,3384	0,2429	-0,1776					-40,3189	0,8937	0,9754	59,076	48
	8,2144	5,7246	4,0960	-3,0108					-1,0600	44,8048			
5	0,5735	0,2506	0,0361	-0,0613	-0,0329	0,1300			-35,1236	0,8958	0,9702	64,95	48
	7,6905	4,2509	0,7271	-1,2418	-0,5644	1,7521			-0,8197	40,8541			
7	0,4651	0,1309	0,0252	0,0588	0,1414	0,1836	0,0952	-0,2122	-16,0169	0,886	0,9766	57,59	48
	6,7120	2,9333	0,5097	1,3223	3,2063	3,7413	2,155	-3,0101	-0,4171	45,5758			

D.W.'s were not available
t values are given below parameter-estimates

2.6 THE ANALYSIS OF AUTOCORRELATION

As was indicated in the previous section, the Labour, Import and Wage equations had significant autocorrelations. This results in unbiased but inefficient estimates in the sense that the variances of the parameter estimates are under estimated.*

Under the assumption that the autocorrelation followed a first order scheme, that is, in equations of the form

$$Y = \beta X_t + u_t$$

$$u_t = \rho u_{t-1} + v_t$$

where $v_t \sim N(0,\sigma_v^2)$ independently

a Cochrane-Orcutt* iterative procedure (employed by the computer package AUTO) was used to make the necessary adjustments. Certain computational difficulties arose in the case of the Wage and Import equations because of a high estimated $\hat{\rho}$ (estimate of ρ - see above) resulting in singularities in matrices that must be inverted in the procedure. In the case of the Wage equation this was circumvented by suppressing the constant term. The following results were obtained

$$L = 23226{,}3879 + 0{,}4365.KNA_{-1} + 1{,}4166\ GNA_{-1}$$
$$(6{,}5949) \qquad (4{,}8323) \qquad (2{,}2043)$$
$$-126{,}6208.S1 - 37{,}4882.S4$$
$$(-1{,}7627) \qquad (-0{,}7208)$$

$R^2 = 0{,}9982$ S.E. = 208,1902 D.W. = 1,6833

d.o.f. = 52 $\hat{\rho} = 0{,}9741$

---

*See Appendix A for details.

$W \quad = 0{,}5435 \cdot Y_{-1}$

$\quad\quad (98{,}2257)$

$R^2 \quad = 0{,}9792 \quad S.E. = 54{,}6537 \quad D.W. = 2{,}1683$

$d.o.f. = 56 \quad \hat{\rho} = 0{,}525$

In the labour equation autocorrelation of the transformed variables is indeterminate according to the cut off points (see Section 2.4), but the Durbin-Watson has improved from 0,44 to 1,68. The coefficient of S4 has become non-significant, however, and it seemed evident that a study of the multicollinearity in the equation would be of value before any judgement could be made on which equation was more reasonable.

The Durbin-Watson statistic in the Wage equation indicates that no autocorrelation is present. The fact that the equation is constrained to pass through the origin, however, has probably weakened its structure as the constant term was significant in the original O.L.S. estimation.

2.7 THE ANALYSIS OF MULTICOLLINEARITY

A number of the equations had two or more explanatory variables which one would expect to be highly collinear, on the grounds that they are stock or flow variables, with similar growth rates. The main consequences of multicollinearity are as follows:

1. The precision of estimation falls so it becomes very difficult to disentangle the relative influences of the various explanatory variables (O.L.S. will frequently give the wrong sign to some variables). These estimates will have large errors and the sampling variances of the coefficients will be very large.

2. Large variances will in turn imply dropping variables incorrectly from the model because of low t-statistic values which are not significantly different from zero, whereas the true situation may well have been that the set of sample data had not been able to pick up the effect of the variable and that it did in fact exist.

An explicit discussion of the technique of ridge regression and characteristic root regression used to adjust for the problem of multicollinearity is given in Appendix A. The three equations that one might expect to exhibit multicollinearity are those for Labour, Inventories and Imports. For ease of presentation and continuity their analysis will be treated separately. The following points are relevant regarding the analysis below.

(i) No distribution theory has up to this point been developed which enables us to test the significance of the estimates in Ridge regression. Multicollinearity, however, results in large variances and small t-values so that the original t's are unreliable. It is often the case (for example with a small t-statistic) that ridge will affect the coefficient significantly, even possibly changing its sign, to yield a correct economic interpretation whereas before this was not obtained.

(ii) The R^2 calculated for the Ridge estimates uses the definition of R^2, that is

$$R^2 = \frac{\hat{\beta}'X'Y - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)^2}{Y'Y - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)^2} \qquad \text{(see Appendix A)}$$

In the case of Ridge regression this does not measure the percentage explanation of the explanatory variables as it does in O.L.S.

$$\frac{\text{Explained Sum of squares}}{\text{Total Sum of squares}} = \frac{\hat{\beta}'X'Y + K\hat{\beta}\hat{\beta}' - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)^2}{Y'Y - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)^2}$$

gives the usual interpretation of R^2. The R^2 given is thus biased downwards if one is to interpret it as above.

(iii) In the case of Ridge regression (as developed by Hoerl and Kennard (1970)) no optimal k exists and a k has to be selected on the basis of the stability of the parameter estimates.

A Ridge trace is plotted (see Appendix A) for each equation system - that is, a two dimensional plot of the parameter estimates against k. Stability will then be evident from this plot.

2.7.1 MULTICOLLINEARITY IN THE LABOUR EQUATION

The following correlation matrix was obtained:

	L	KNA_{-1}	GNA_{-1}	S1	S4
L	1				
KNA_{-1}	0,9858	1			
GNA_{-1}	0,9932	0,9828	1		
S1	0,0216	0,0253	0,02626	1	
S4	-0,0198	-0,0244	0,03635	-0,333	1

A near singularity occurs in the X'X matrix because of the high correlation between KNA_{-1} and GNA_{-1} (see Appendix A). Using the technique of Ridge regression a Ridge trace is obtained (see following page). Equations are given for $k = 0{,}01$ and $k = 0{,}1$; $k = 0{,}01$ yields

$$L = 0{,}2279.KNA_{-1} + 5{,}8103.GNA_{-1} - 502{,}4078.S1 - 207{,}543.S4 + 15912{,}8701$$

$R^2 = 0{,}9834$

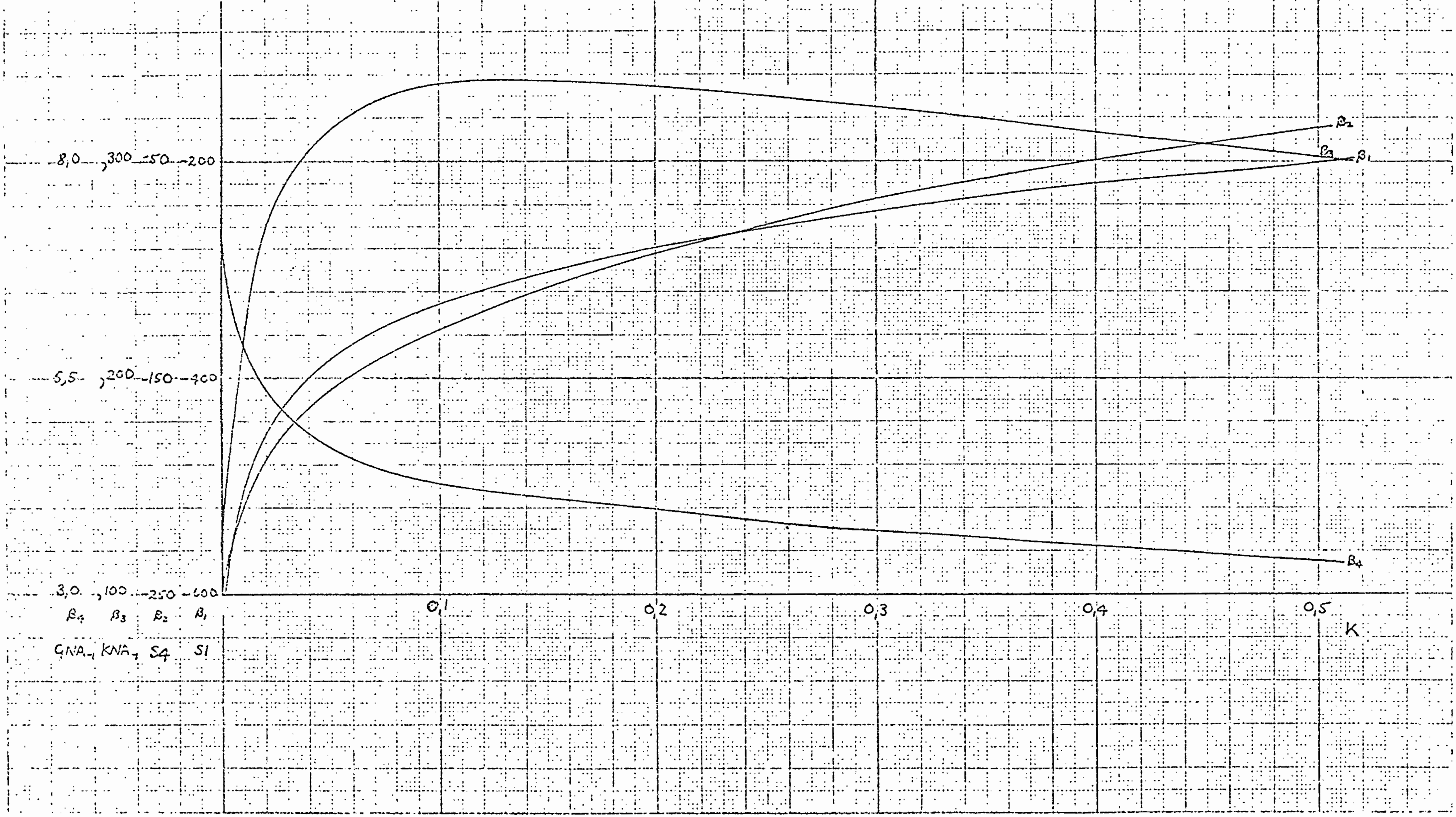
$k = 0{,}1$

$$L = 16765{,}22 + 0{,}3355.KNA_{-1} + 4{,}3025.GNA_{-1} - 332{,}64.S1 - 127{,}37.S4$$

$R^2 = 0{,}9359$

The Ridge trace suggests that the system is reasonably stable at $k = 0{,}1$.

RIDGE TRACE — LABOUR EQUATION



Of possibly greater value (especially regarding the form of the data space) than a Ridge interpretation is consideration of the eigenvalues and eigenvectors of the (XY)'XY matrix and the characteristic root estimates. (See Appendix A)

The eigenvalues of the (XY)'(XY) matrix - λ_i; $i = 1,2,\ldots,p$; $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ are given below with corresponding γ_{oi} (see Appendix A).

i	1	2	3	4	5
λ_i	2,9749	1,3333	0,6684	0,0185	0,0049
γ_{oi}	-0,5780	-0,00394	-0,02779	0,41109	-0,70439

The data space is thus basically three dimensional with very little variation in the direction of vectors 4 and 5. The degree of correlation with y is provided by the value of γ_{oi} (see Appendix A). It can be seen, therefore, that although singularities exist they are predictive singularities (that is λ_4 and λ_5 are small but γ_{04} and γ_{05} are large). For clarification a geometric picture is presented of the data space in terms of the eigenvalues and eigenvectors above. (See Figure 2.3 below.)

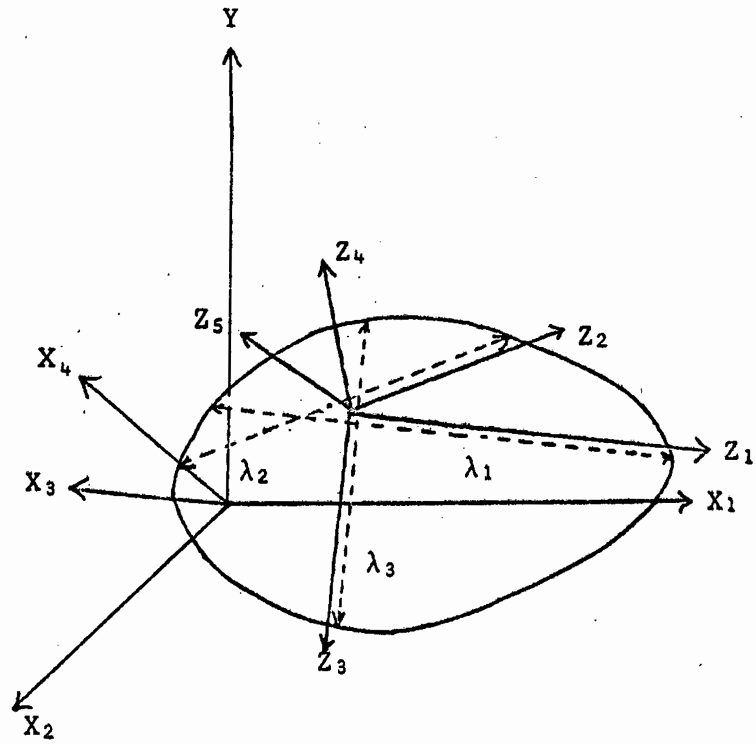


Figure 2.3 Geometric picture of data space (Labour equation)

X_i - orthogonal axes

Z_i - transformed orthogonal axes (in direction of γ_j)

γ_4 and γ_5 lie close to the Y axis, (their projection on the Y axis is large) but variation in the direction of γ_4 and γ_5 is small hence small λ_4 and λ_5.

Computation of Modified Least Square (MLS) estimates is of interest for the sake of comparison, taking cognizance of changes in sums of variance and residual sum of squares. For example, MLS (eliminating eigenvalues 2 and 3) yields

$$L = 15739{,}2231 + 0{,}1241.KNA_{-1} + 6{,}9623.GNA_{-1} - 619{,}7009.S1 - 253{,}1004.S4$$

The change in the coefficients is non-significant even though the data space has been reduced by 2 dimensions.

The sum of variances $= \sum_{i=1,4,5} \frac{1}{\lambda_i}$ (see Appendix A)

$= \sigma^2 259,97$

Note: O.L.S. sum of variances $= \sum_{i=1}^{5} \frac{1}{\lambda_i} = \sigma^2 262,22$

Residual sum of squares $= \eta^2 (\sum_{j=1,4,5}^{5} \gamma_{oj}^2/\lambda_j)^{-1}$ (see Appendix A η^2 a constant)

$= \eta^2 . 0,903$

Note: O.L.S. residual sum of squares $= \eta^2 (\sum_{j=1}^{5} \gamma_{oj}^2/\lambda_j)^{-1}$

$= \eta^2 . 0,0043$

The residual sum of squares has increased dramatically but the change in the sums of variance is insignificant.

Compare this now with an MLS solution with eigenvalue 5 removed:

$L = 1,5157.KNA_{-1} - 8,1463.GNA_{-1} + 859,24.S1 + 275,27.S4 + 17371,1$

Because of the removal of a vector which was closely correlated to Y ($\gamma_{05} = -0,7$) the change has been quite marked, with:

Sum of variances $= \sum_{i=1}^{4} \frac{1}{\lambda_i}$

$= \sigma^2 58,14$

Residual sum of squares $= (\sum_{j=1}^{4} \gamma_{oj}^2/\lambda_j)^{-1}$

$= \eta^2 \ 0,105$

Unfortunately we were unable to test the forecasting power of these equations because of the unavailability of quarterly data on income in the non-agricultural sector.

2.7.2 MULTICOLLINEARITY IN THE IMPORT EQUATION

The correlation matrix was as follows:

	Imp	Imp_{-1}	Gde
Imp	1		
Imp_{-1}	0,977	1	
Gde	0,946	0,945	1

There is thus large correlation between the independent variables (0,945).

The Ridge trace is shown on the following page.

k = 0,01 yielded

Imp = $0{,}776 \cdot Imp_{-1} + 0{,}0685 . Gde - 25{,}465$

$R^2 = 0{,}9512$

k = 0,1

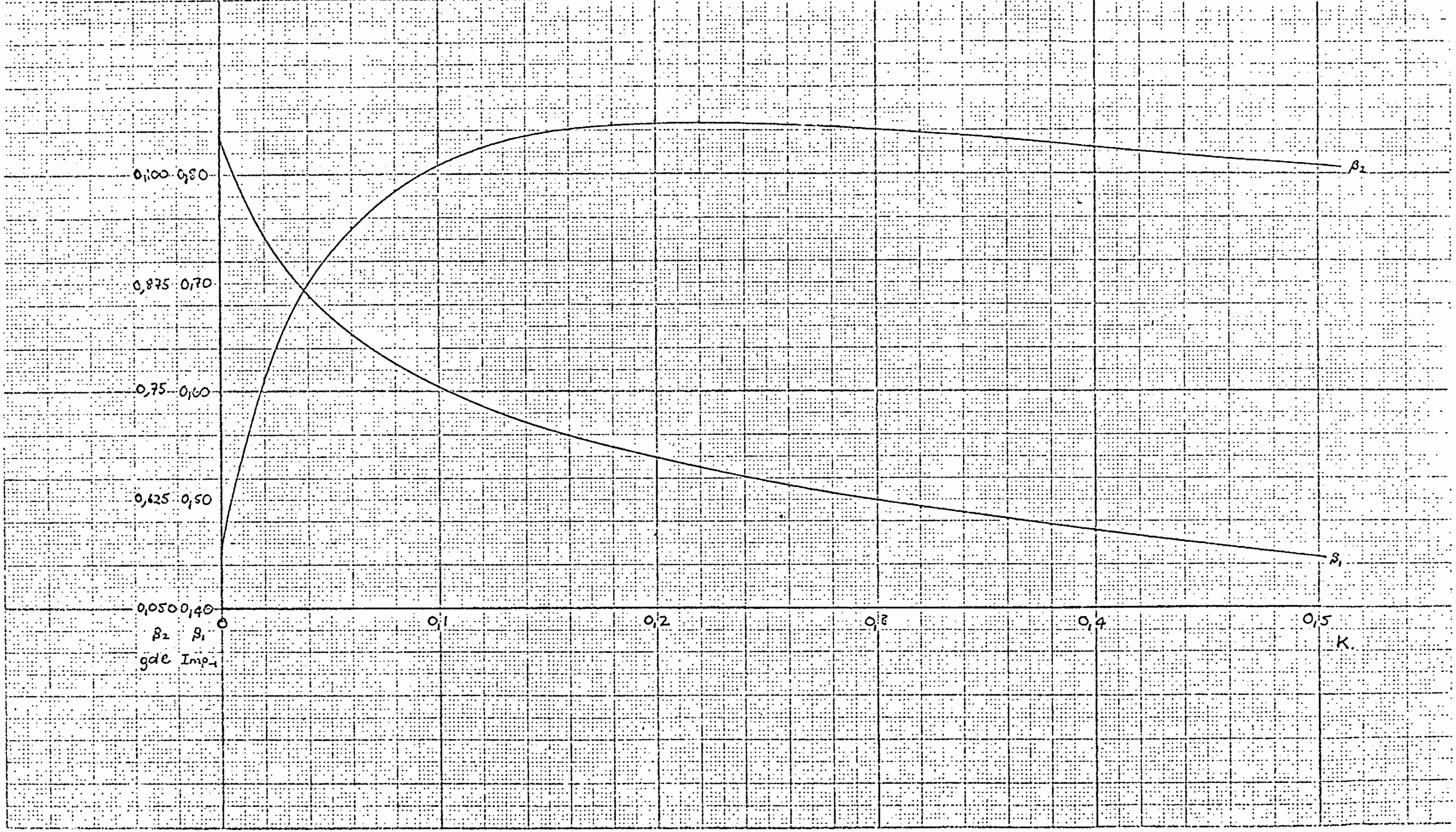
Imp = $0{,}6044 \cdot Imp_{-1} + 0{,}111 \cdot Gde - 2{,}9036$

$R^2 = 0{,}9035$

Inspection of the Ridge trace shows that adequate stabilization is achieved with k = 0,1.

The eigenvalue , y - component eigenvector spectrum is given below:

RIDGE TRACE - IMPORT EQUATION



i	1	2	3
λ_i	2,91215	0,06487	0,0229
γ_{oi}	0,5795	0,4008	-0,7096

The singularity that exists is a predictive singularity and therefore we could not expect to improve our estimates with MLS.

2.7.3 MULTICOLLINEARITY IN THE INVENTORIES EQUATION

The correlation matrix is as follows:

	S	S4	Exp	Y_{-1}
S	1			
S4	0,02820	1		
Exp	0,83689	-0,04499	1	
Y_{-1}	0,97737	0,03009	0,88523	1

Of particular concern therefore would be the high correlation between Exp and Y_{-1} (0,88523).*

As before, observance of the Ridge trace is indicative of the value of k for which stability of the estimates is reached. Estimates for $k = 0{,}01$ and $k = 0{,}1$ are given below; estimates with $k = 0{,}1$ exhibit adequate stability.

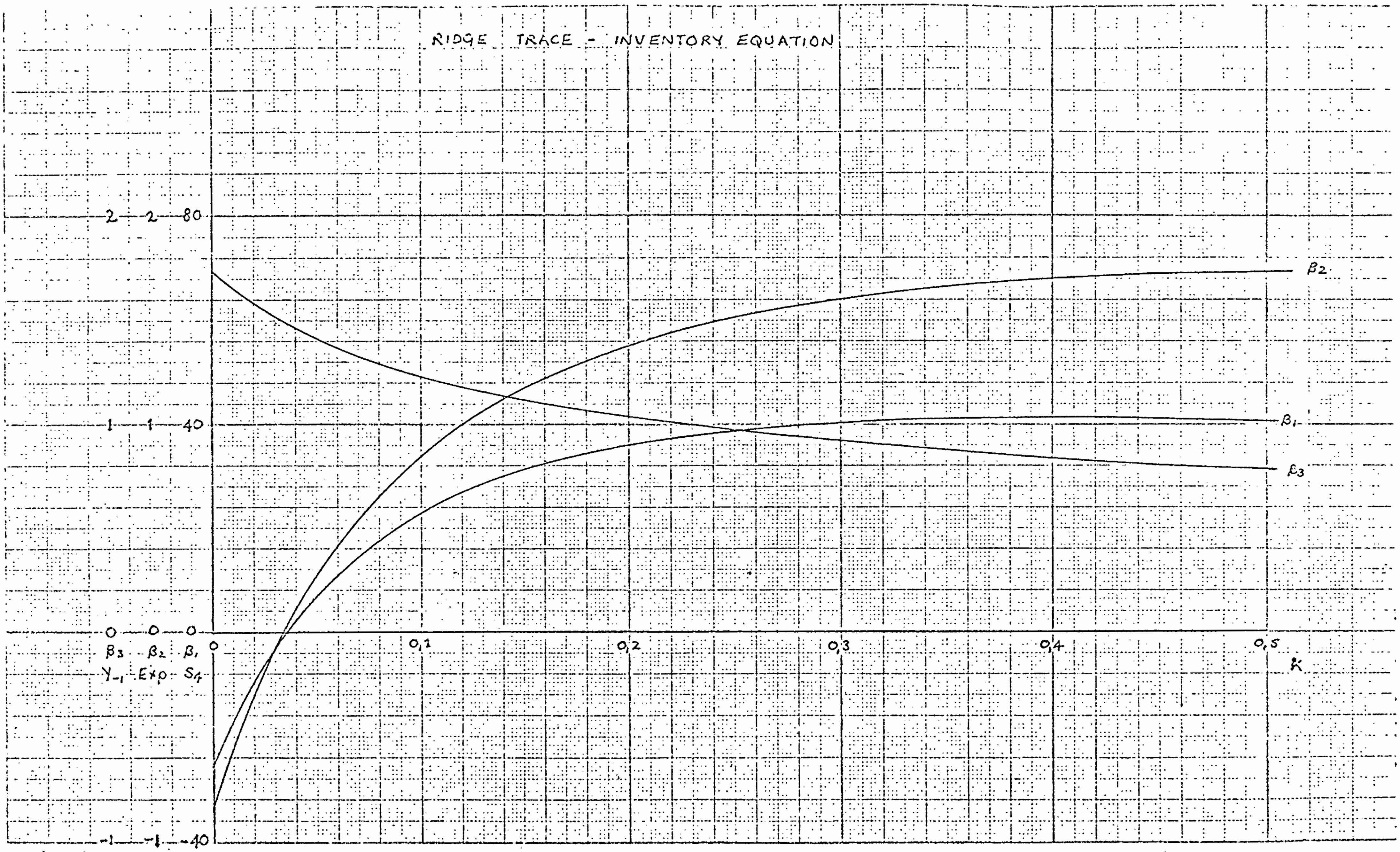
$k = 0{,}01$

$S = -565{,}9230 + 1{,}6472.Y_{-1} - 0{,}8732.\text{Exp} - 26{,}4961.S4$

$R^2 = 0{,}9269$

---

* As with the equation above MLS was inapplicable because no non-predictive singularities existed.


RIDGE TRACE - INVENTORY EQUATION
2 2 80
1 1 40
0 0 0
β_3 β_2 β_1
Y_{-1} Exp S_1
-1 -1 -40
0
0,1
0,2
0,3
0,4
0,5
k
β_2
β_1
β_3

$k = 0,1$

$S = -478,8607 + 1,2347.Y_{-1} + 0,8540.Exp + 23,1203.S4$

$R^2 = 0,8681$

2.7.4 CONCLUSIONS FOR RIDGE ADJUSTMENT

For the Labour equation the coefficients of GNA_{-1}, S1 and S4 exhibit quite large percentage decreases, and KNA_{-1} a large percentage increase. There are no sign changes, however, and the specification seems to hold up well to the adjustment. The Import equation adjustment yields a much larger coefficient for Gde and a smaller coefficient for Imp_{-1} - as with the above no sign changes occur. The most marked changes take place in the Inventory equation where the coefficients of Exp and S4 change from negative to positive; and the coefficient of Y_{-1} becomes smaller. As discussed in Appendix A, R^2 shows the expected decrease in each case. The above results should be considered in the light of the theoretical discussion of Ridge estimation in Appendix A, with special reference to the fact that we know that increasing k will result (for large k) in decreasing absolute values for the parameters. We must be careful, therefore, to consider a k that is beneficial in the sense that it adjusts for multicollinearity and does not swamp the coefficient. Bearing this in mind, it is noted that in the inventory equation neither Exp or S4 change sign with $k = 0,01$. Consideration of the simulation study (Appendix A) would indicate that if the equation was yielding estimates of the wrong sign these would change to the correct sign very quickly ($k = 0{,}01$). Therefore it is doubtful whether the inventory equation is in fact mis-specified - but rather that we have taken a k that has gone beyond the minimum mean square error point (see Appendix A).

The purpose of this equation system was to obtain forecasts of real sector variables in the South African economy; forecasting accuracy would also enable us to discriminate between estimates (O.L.S. and Ridge) for the purpose of a final specification.

2.8 FORECASTING PERFORMANCE

Forecasts of the real sector variables and the level of prices were carried out for the four quarters of 1975. All figures are constant 1975 prices. Forecasts are carried out by the simultaneous solution of the structural equations given the exogenous and predetermined endogenous variables, for example, to calculate C we need Disposable Income which in turn requires the calculation of Wages and Personal Taxes.

Note:

(i) Actual figures on Personal Taxes and Wages for the specified period have not been released by the Reserve Bank.

(ii) It was not possible to forecast labour in the non-agricultural sector because data on income in the non-agricultural sector has not been released by the Reserve Bank.

FORECASTS FOR 1975 ON QUARTERLY BASIS

All figures R million (1970 prices)	Forecast	Actual	% error	Forecast	Actual	% error	Forecast	Actual	% error	Forecast	Actual	% error
	1975 - 1st Quarter			1975 - 2nd Quarter			1975 - 3rd Quarter			1975 - 4th Quarter		
C (O.L.S)	2049	1960	4,5	2040	2065	-1,2	2100	2072	1,3	2217	2172	2,1
I (O.L.S)	1145	965	18,6	1073	1102	-2,6	1154	1082	8,9	1133	1146	-1,1
Imp (O.L.S)	1165	1148	1,5	1159	1092	0,61	1104	1069	8,9	1084	1102	-1,6
Imp (Ridge, k=0,1)	1093	1148	-4,8	1114	1092	2,0	1064	1069	-0,5	1053	1102	-4,4
Exp (O.L.S.)	1009	1055	-4,3	1046	983	6,4	1053	1014	3,8	1033	947	9,1
P_eTAX (O.L.S.)	221			147			222			169		
W (O.L.S.)	1995			1881			2001			1973		
S (O.L.S)	5040	5205	-3,2	5408	5426	-0,3	5405	5484	-1,4	5365	5499	-2,4
S (Ridge, k=0,1)	5150	5205	-1,1	5361	5426	-1,2	5201	5484	-5,2	5070	5499	-7,8
ΔS (O.L.S)	-140	25	-760	368	221	66,5	-3	58	-1,1	-40	15	-433,3
ΔS (Ridge, k=0,1)	-30	25	-220	211	221	-4,5	-80	58	-237,9	-211	15	-1513,3
ΔP (O.L.S)	5,2	6,6	-212	4,7	4,8	-1,3	5,4	7,3	-26,0	10,7	8,8	21,6

(iii) The O.L.S. equations used for forecasting are those marked (**) in the O.L.S. equation presentation.

(iv) Although we estimate levels of inventory from our equation system, forecasts on changes in inventories are given because that is how they appear in the National Accounts.

2.9 CONCLUSIONS

The model has, with the exception of ΔS (change in inventories), succeeded in producing a reasonably accurate picture of the performance of the economy in 1975. Similar difficulties with the forecasting of inventory changes were experienced by de Wet and Dreyer (1977), as a result apparently of the fact that the "Inventory changes are probably the poorest in the entire set of national account" - (de Wet and Dreyer (1977)).

Of particular interest was the forecasting performance of the Ridge equations especially as regards a comparison with their O.L.S. equivalents. In both the case of imports and inventories neither were clearly superior, and it appears that their performance will have to be more fully examined before a choice is made between the two.

The approximation to the workings of the real sector of an economy by a system of linear equations is, however, not realistic. We have good a priori reasons for expecting the derivatives and the second derivatives of a number of the relations discussed to be variable. The greatest potential of the linear system is to provide consistent forecasts of the time paths of the variables under consideration. We cannot attempt to give anything more than a crude economic justification for the working of such a model - for the most part we have regarded the economic system as a "black box" and relied on empirical regularities in the data for the success of such a specification.

CHAPTER THREE

THE THEORY OF CONTINUOUS TIME MODEL ESTIMATION

3.1 INTRODUCTION

What was 'genuine' forecasting performance?

The detailed analysis of the linear model in Chapter 2 had as its primary target the establishment of a flexible forecasting system. The use of economic theory was limited and although the explanation of variation was extremely satisfactory it gave inadequate insight into the inter-relationships between variables and their rates of change. Stimulated by work done by Wymer (1974) and Hurwitz (1977) the author decided to construct a system that took the form of simultaneous partial equilibrium equations, where extensive use may be made of economic theory. Their applicability and form will be discussed in detail in the next chapter. These partial equilibrium equations are a specific case of a wide range of mathematical representations of stochastic phenomena, known as "continuous time systems".

The use of continuous simultaneous equation systems to describe economic behaviour stemmed from the pioneering work by Mann and Wald on difference equation systems (1943) and has recently been developed extensively by Sargan (1974) Bergstrom (1966) Wymer (1972) and P.C.B. Phillips (1972).

The limitations imposed on this analysis result from the problem of making statistical inferences from discrete data about the parameters of models whose variables are continuous functions of time. In this chapter we deal with the development of the approximate discrete model of C.R. Wymer developed to estimate an r^{th} order differential equation system with a white noise error process. It will initially be of interest to discuss a few concepts relating to the theory of stationary random functions and the mathematical idealization known as "white noise".

3.2 STATIONARY RANDOM FUNCTIONS

3.2.1 DEFINITION

A random function $u(t)$ will be called stationary if all finite distributions function of the form

$F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n) = P\{u(t_1)<x_1, u(t_2)<x_2,\ldots,u(t_n)<x_n\}$

$t_i \in T$, T the real line

$i = 1,\ldots,n$

which define $u(t)$, remain the same if the whole group of points $t_1, t_2,\ldots,t_n$ is shifted along the time axis i.e.

$F_{t_1+\tau,t_2+\tau,\ldots,t_n+\tau}(x_1,x_2,\ldots,x_n) = F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$ for any τ.

From the above immediately follows that all one-dimensional distribution functions of the above must be identical, i.e. $F_t(x)$ cannot depend on t. $\therefore$ $E(u(t)) = m$ (constant). We will in fact without loss of generality consider from this point functions of the form $u(t) - m$, which have mean zero. Two dimensional distribution functions can from the above only depend on the difference $t_1 - t_2$.

3.2.2 THE CORRELATION FUNCTION

We define $E(u(t)u(s)) = B(t,s)$.

The function $B(t,s)$ is called the correlation function of $u(t)$. If $u(t)$ is stationary then $B(t,s)$ depends only on the difference $t-s$ i.e.

$$E(u(t)u(s)) = B(t-s)$$
$$= B(\tau), \qquad \tau = t-s.$$

We may assume now that $B(\tau)$ is continuous. In fact a sufficient condition for $B(\tau)$ to be continuous for all τ is that $B(\tau)$ is continuous at the point $\tau = 0$. (Yaglom (1973) P.22).

As will become clear, of special interest to us is the particular case where $B(\tau)$ falls off so rapidly as $|\tau| \to \infty$ that:

$$\int_{-\infty}^{\infty} |B(\tau)|d\tau < \infty \qquad (3.2.2.1).$$

3.2.3 THE SPECTRAL REPRESENTATION

Any stationary process may be given the following spectral representation - Yaglom (1973):

$$u(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda) \qquad (3.2.3.1)$$

where λ is a real constant, and $Z(\lambda)$ is a random point function with zero mean and uncorrelated increments (3.4.2). In the case where (3.2.2.1) applies we may write the correlation function as

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} dF(\lambda) \qquad (3.2.3.2)$$

where $F(\lambda)$ is the "spectral distribution function" of $u(t)$ and is real, non-decreasing, and bounded and where

$f(\lambda) = F'(\lambda)$ is the "spectral density"of $u(t)$.

From (3.2.3.2) we have immediately that:

$$B(0) = E(u(t))^2 = \int_{-\infty}^{\infty} dF(\lambda) = F(\infty) - F(-\infty) < \infty \quad \text{(since F is bounded)} \quad (3.2.3.3)$$

We can write (3.2.3.2) as the Fourier integral:

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\lambda) d\lambda$$

The inverse Fourier integral is thus:

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda\tau} B(\tau) d\tau$$

Considering the case (3.2.2.1) where the correlation falls off rapidly we may take

$$B(\tau) = Ce^{-\alpha|\tau|}$$

$$C > 0, \ \alpha > 0$$

thus $$f(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} Ce^{-\alpha|\tau|} d\tau$$

$$= \frac{C}{\pi} \; \frac{\alpha}{\alpha^2+\lambda^2} \qquad \text{(Yaglom (1963) p.62)}$$

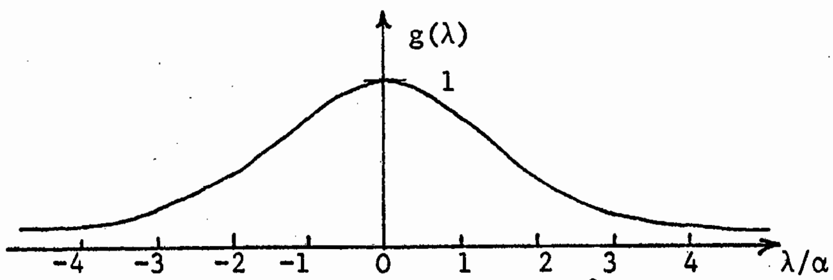


Figure 3.1 Graph of the function $g(\lambda) = \frac{\alpha^2}{\alpha^2+\lambda^2}$.

The quicker the fall off in correlation the larger α, which will infer that λ is small compared to α . In such a situation the spectral density is practically constant and approximately equal to $f(0) = f_0$. Moreover if only a small range of λ is under consideration it is reasonable to assume that $f(\lambda) = f_0 = \text{constant}$. However, strictly speaking, no random process $u(t)$ can ever exist with a constant spectral density, for otherwise the quantity

$$E(u(t))^2 = B(0) = \int_{-\infty}^{\infty} f(\lambda)d\lambda = \infty$$

and we know that $B(0)$ is bounded (3.2.3.3).

3.3 "WHITE NOISE"

The concept of a random process with a constant spectral density is a useful mathematical idealization but cannot be rigorously defined (except with reference to complex topological spaces - Yaglom (1973)). It is often referred to as "white noise" (Jazwinski (1970)).

3.4 THE INTEGRAL OF A RANDOM PROCESS WITH CONSTANT SPECTRAL DENSITY

Under the assumption that $u(t)$ is a random process with constant spectral density, we consider the integral

$$\zeta(t) = \int_0^t u(s)\,ds \quad ; \quad u(t) = \zeta'(t)$$

$\zeta(t)$ will no longer be stationary since

$$B(0) = E[\zeta(t)]^2 = \int_0^t\int_0^t E[u(s)u(s')]\,ds\,ds'$$

$$= \int_0^t \int_0^t B(s-s')\,ds\,ds' .$$

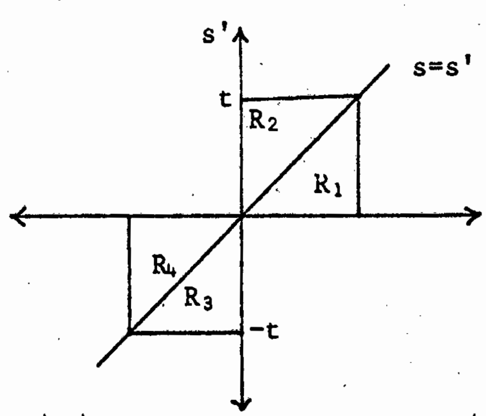
We have taken $B(s) = Ce^{-\alpha|s|}$

$$\therefore \quad E[\zeta(t)]^2 = \int_0^t \int_0^t Ce^{-\alpha|s-s'|}\,ds\,ds' .$$

For $t > 0$

$$= C\int\int_{R_1} e^{-\alpha(s-s')}\,ds\,ds' + C\int\int_{R_2} e^{\alpha(s-s')}\,ds\,ds'$$

where R_1 and R_2 are the areas defined below.



$$= C\int_0^t \int_{s'}^t e^{-\alpha(s-s')}\,ds\,ds' + C\int_0^t \int_0^s e^{\alpha(s-s')}\,ds\,ds'$$

$$= \frac{C}{\alpha}\left[2t + \frac{2}{\alpha}e^{-\alpha t} - \frac{2}{\alpha}\right]$$

letting $C \to \infty$ and $\alpha \to \infty$ such that $2\frac{C}{\alpha}$ = constant = c (since $f_0=\frac{C}{\pi\alpha}$ is constant)

$= ct.$

Similarly for $t < 0$

$$E[\zeta(t)]^2 = \iint_{R_3} e^{-\alpha(s-s')}ds\,ds' + \iint_{R_4} e^{\alpha(s-s')}ds\,ds'$$

$$= \int_t^0\int_{s'}^0 e^{-\alpha(s-s')}ds\,ds' + \int_t^0\int_t^{s'} e^{\alpha(s-s')}ds\,ds'$$

$$= \frac{C}{\alpha}\left[-2t + \frac{2}{\alpha}e^{\alpha t} - \frac{2}{\alpha}\right]$$

$$= -ct$$

$\therefore E\left[\zeta(t)\right]^2 = c|t|.$

Stationarity required that:

$E(u(t)u(s)) = B(t-s)$ is a function of $t-s$; and hence that $E(u(t)u(t))$ be independent of time.

Therefore $\zeta(t)$ is not stationary.

We can also show that $\zeta(t)$ has orthogonal increments i.e.

$$E\left[\zeta(t_2) - \zeta(t_1)\right]\left[\zeta(t_4) - (t_3)\right] = 0$$

if $t_1 < t_2 \leqslant t_3 < t_4$

This follows by considering two closed intervals $[t_1,t_2],[t_2,t_3]$ with common t, (The general case can be reduced to this case, see Yaglom (1973)) with $0 < t_1 < t_2 < t_3$.

We have:

$$E[\zeta(t_3)-\zeta(t_1)]^2 = E[\zeta(t_3)-\zeta(t_2)]^2+E[\zeta(t_2)-\zeta(t_1)]^2+2E[\zeta(t_3)-\zeta(t_2)][\zeta(t_2)-\zeta(t_1)]$$

$$\therefore \quad c(t_3-t_1) = c(t_3-t_2) + c(t_2-t_1) + 2E[\zeta(t_3)-\zeta(t_2)][\zeta(t_2)-\zeta(t_1)]$$

So we have $E[\zeta(t_3)-\zeta(t_2)][\zeta(t_2)-\zeta(t_1)] = 0$

The above process $\zeta(t)$ with the above properties is said to be an "homogenous random process with uncorrelated increments".

In the case where $\zeta(t)$ is a vector with constant "spectral density matrix" Ω the above generalizes to:

$$E[\zeta(t)][\zeta(t)]' = \Omega|t| \qquad \forall t \tag{3.4.1}$$

$$E[\zeta(t_2) - \zeta(t_1)][\zeta(t_4) - \zeta(t_3)]' = 0 \tag{3.4.2}$$

$$t_1 < t_2 \leq t_3 < t_4 \ .$$

3.5 STOCHASTIC DIFFERENTIAL EQUATION SYSTEMS

3.5.1 THE BASIC DIFFERENTIAL EQUATION MODEL

Consider the stochastic differential equation system of the form (Wymer(1972)):

$$D^r y^*(t) = \sum_{k=1}^{r} A_k^* D^{k-1} y^*(t) + B^* z(t) + u^*(t) \tag{3.5.1.1}$$

where D is a stochastic differential equation operator

$y^*(t)$ is a vector of n endogenous variables

$z^*(t)$ is a vector of m exogenous variables

$u^*(t)$ is a vector of disturbances.

A_k^* are matrices of order n

B^* is a matrix of dimension $n \times m$.

This may be reduced to the first order system (Hirsch and Smale (1974):

$$Dy(t) = Ay(t) + Bz(t) + u(t)$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_r(t) \end{bmatrix} ,$$

$$A = \begin{bmatrix} 0 & . & I \\ . & . & . \\ A_1 & . & A_2 \dots A_r \end{bmatrix}$$

$$u(t) = \begin{bmatrix} 0 \\ u^*(t) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ B^* \end{bmatrix}$$

with $\quad Dy_i(t) = y_{i+1}(t) \qquad i = 1.....r-1$

$$Dy_r(t) = \sum_{k=1}^{r} A_k^{\theta} y_k(t) + B^* z^{\theta}(t) + u^*(t)$$

In particular we consider an r^{th} order system reduced to a first order system of the form:

$$Dy(t) = A(\theta)y(t) + B(\theta)z(t) + u(t) \qquad (3.5.1.2)$$

where: θ is a vector of model parameters, A and B are matrices with elements that are differentiable functions of θ.

Wymer (1972) makes the assumption that u(t) is a stationary vector process with constant spectral density matrix Ω. i.e. u(t) is a *white noise process*.

The system (3.5.1.2) can be written equivalently as

$$dy(t) = A(\theta)y(t)dt + B(\theta)z(t)dt + u(t)dt. \qquad (3.5.1.3)$$

We define:

$$\zeta(t) = \int_0^t u(s)ds \ .$$

From the above discussion we have that $\zeta(t)$ is an homogenous process with uncorrelated increments.

Since u(t) is not rigorously defined we replace u(t)dt in

(3.5.1.3) by $d\zeta(t)$, the mean square differential of $\zeta(t)$ to yield:

$$dy(t) = A(\theta)y(t)dt + B(\theta)z(t)dt + d\zeta(t) \qquad (3.5.1.4)$$

This has the solution

$$y(t) = e^{At}y(0) + \int_0^t e^{A(t-\theta)}Bz(\theta)d\theta + \int_0^t e^{A(t-\theta)}d\zeta(\theta) \qquad (3.5.1.5)$$

3.5.2 THE EXACT DISCRETE ANALOGUE

Given a set of observations $y(t)$ and $z(t)$ at points $t = \tau\delta$

$\tau = 1,2,\ldots,n$

δ the observation interval,

we construct an exact discrete analogue of (3.5.1.5) in the sense that the two solutions are identical on the discrete time set $\{t = \tau\delta;\ \tau = 1,2,\ldots,n\}$. The exact discrete model is derived by integrating (3.5.1.4) over the interval $(\tau\delta-\delta,\tau\delta)$ where $t = \tau\delta$ to obtain

$$y(t) = e^{\delta A}y(t-\delta) + \int_0^\delta e^{As}Bz(t-s)ds + \int_0^\delta e^{AS}d\zeta(t-s) \qquad (3.5.2.1)$$

where the error process:

$$w_t = \int_0^\delta e^{As}d\zeta(t-s) \qquad (3.5.2.2)$$

is well defined as a Stieltjies Integral and

$$E(w_t w_t') = E\left\{\left[\int_t^{t-\delta} e^{(t-s)A}d\zeta(s)\right]\left[\int_t^{t-\delta} e^{(t-s)A}d\zeta(s)\right]'\right\}$$

$$= E\left\{\left[\lim_{n\to\infty}\sum_{i=1}^n e^{\frac{\delta i}{n}A}\left(\zeta\left(t-\frac{\delta i}{n}\right)-\zeta\left(t-\frac{\delta(i-1)}{n}\right)\right)\right]\left[\lim_{n\to\infty}\sum_{i=1}^n e^{\frac{\delta i}{n}A}\left(\zeta\left(t-\frac{\delta i}{n}\right)-\zeta\left(t-\frac{\delta(i-1)}{n}\right)\right)\right]'\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^n e^{\frac{\delta i}{n}A}E\left[\left(\zeta\left(t-\frac{\delta i}{n}\right)-\zeta\left(t-\frac{\delta(i-1)}{n}\right)\right)\left(\zeta\left(t-\frac{\delta i}{n}\right)-\zeta\left(t-\frac{\delta(i-1)}{n}\right)\right)'\right]\sum_{i=1}^n e^{\frac{\delta i}{n}A'}\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n} e^{\frac{\delta i}{n}A}\,\Omega\left|\frac{\delta}{n}\right|\sum_{i=1}^{n} e^{\frac{\delta i A'}{n}}\right\} \qquad \text{(By 3.4.1)}$$

$$= \int_0^{\delta} e^{sA}\Omega e^{sA'}ds \qquad (3.5.2.3)$$

(by definition of a Stieltjies Integral)

Therefore even if Ω is diagonal the w_t are not independent.

If we consider $E(w_t w_{t-1})$ we get as our centre term

$$E\left[\left(\zeta\left(t-\frac{\delta i}{n}\right)-\zeta\left(t-\frac{\delta(i-1)}{n}\right)\right)\left(\left(t-\delta-\frac{\delta i}{n}\right)-\zeta\left(t-\delta-\frac{\delta(i-1)}{n}\right)\right)'\right] = 0$$

(by 3.4.2)

and similarly $E(w_t w_{t-r}) = 0, \ \forall r$

The w_t are thus serially independent.

For further reference we may rewrite (3.5.2.1) as

$$y_t - e^{\delta A}y_{t-1} = \psi_t + w_t \qquad (3.5.2.4)$$

where $y(\tau\delta) = y_t, z(\tau\delta) = z_t, w(\tau\delta) = w_t$

and $\int_0^{\delta} e^{sA}Bz(\tau\delta-s)ds = \psi_t$.

3.5.3 THE MIXED STOCK FLOW MODEL

The above is only relevant to models with variables that are observable at a point in time. In a continuous system a flow variable $x(t)$ e.g. income, is not measurable at a point in time but its integral

$$x^0(t) = \int_{t-\delta}^{t} x(s)ds$$

is measurable. (δ observation interval)

$x^0(t)$ = the flow during the interval $x(t-\delta)$ through $x(t)$.

A continuous mixed stock/flow model must thus be integrated over the observation period to give a model defined in terms of measureable flow variables. If we integrate a stock variable such as price $p(t)$ we get: $p^0(t) = \int_{t-\delta}^{t} p(s)ds$

$\hat{=} \delta.\frac{1}{2}[p(t) - p(t-\delta)]$

(trapezium rule).

Integrating (3.5.2.1) to obtain measurable quantities we get:

$$\int_{t-\delta}^{t} y(s)ds = e^{\delta A}\int_{t-\delta}^{t} y(s-\delta)ds + \int_{t-\delta}^{t}\int_{0}^{\delta} e^{As}Bz(\theta-s)dsd\theta$$

$$+ \int_{t-\delta}^{t}\int_{0}^{\delta} e^{As}d\zeta(\theta-s)d\theta$$

written as:

$$y_t^0 = e^{\delta A}y_{t-1}^0 + \int_0^\delta e^{As}Bz^0(t-s)ds + \int_{t-\delta}^{t}\int_0^\delta e^{As}d\zeta(\theta-s)d\theta \quad (3.5.3.1)$$

where y_t^0 is the τ observation of the flow variable y

and $z^0(t) = \delta Mz_{(t)}$; $M = \frac{1}{2}(1+L)$, (where L is the lag operator).

We may rewrite this system for further reference as

$$y_t^0 - e^{\delta A}y_{t-1}^0 = \psi_t^0 + \xi_t$$

$$\text{where} \quad \psi_t^0 = \int_0^\delta e^{As}Bz^0(t-s)ds \quad (3.5.3.2)$$

$$\xi_t = \int_{t-\delta}^{t}\int_0^\delta e^{As}d\zeta(\theta-s)d\theta.$$

If δ is small e^{As} will be close to unity over the interval of integration. We can therefore make the following approximation (Wymer(1974)) for the error process of (3.5.3.1):

$$\xi_t = \int_{t-\delta}^{t}\int_0^\delta d\zeta(\theta-s)d\theta .$$

The auto covariance generating function of the above is defined as:

$$H(L) = \sum_{-\infty}^{\infty} \phi_k L^k$$

where $\phi_k = E\left\{\xi_t \xi_{t+k\delta}\right\}$ and L is the lag operator.

Therefore:

$$\phi_1 = E\left(\xi_t \xi_{t+\delta}\right)$$

$$= E\left\{\left[\int_{t-\delta}^{t}\int_{s-\delta}^{s} d\zeta(\theta)ds\right]\left[\int_{t}^{t+\delta}\int_{s-\delta}^{s} d\zeta(\theta)ds\right]'\right\}$$

$$=E\left\{\left[\lim_{n\to\infty}\sum_{i=1}^{n}\int_{t-2\delta+\frac{i\delta}{n}}^{t-\delta+\frac{i\delta}{n}} d\zeta(\theta)\left(\left(t-\delta+\frac{i\delta}{n}\right)-\left(t-\delta+\frac{(i-1)}{n}\delta\right)\right)\right]\left[\lim_{n\to\infty}\sum_{i=1}^{n}\int_{t-\delta+\frac{i\delta}{n}}^{t+\frac{i\delta}{n}} d\zeta(\theta)\left(\left(t+\frac{i\delta}{n}\right)-\left(t+\frac{(i-1)}{n}\delta\right)\right)\right]'\right\}$$

$$=E\left\{\lim_{n\to\infty}\left[\frac{\delta}{n}\sum_{i=1}^{n}\int_{t-2\delta+\frac{i\delta}{n}}^{t-\delta+\frac{i\delta}{n}} d\zeta(\theta)\right]\left[\frac{\delta}{n}\sum_{i=1}^{n}\int_{t-\delta+\frac{i\delta}{n}}^{t+\frac{i\delta}{n}} d\zeta(\theta)\right]'\right\}$$

$$=E\left\{\lim_{n\to\infty}\left[\frac{\delta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\zeta\left(t-2\delta+\frac{(i+j)}{n}\delta\right)-\zeta\left(t-2\delta+\left(\frac{i+j-1}{n}\right)\delta\right)\right)\right]\left[\frac{\delta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\zeta\left(t-\delta+\frac{(i+j)}{n}\delta\right)-\zeta\left(t-\delta+\frac{(i+j-1)}{n}\delta\right)\right]'\right\}$$

$$(3.5.3.3)$$

For reference we denote the first square bracket in (3.5.3.3) by ⊛ and the second square bracket by ⊛⊛.

Writing out ⊛ for i=1,...,n we obtain

$$\frac{\delta}{n}\left[\zeta\left(t-2\delta+\frac{2\delta}{n}\right)-\zeta\left(t-2\delta+\frac{(2\delta-1)}{n}\right)+..+\zeta(t-2\delta+\delta)-\zeta\left(t-2\delta+\frac{(n-1)}{n}\delta\right)+\zeta\left(t-2\delta+\frac{(n+1)}{n}\delta\right)-\zeta(t-2\delta+\delta)\right]$$

$$\vdots$$

$$\frac{\delta}{n}\left[\zeta\left(t-2\delta+\frac{(n+1)}{n}\delta\right)-\zeta(t-2\delta+\delta)+...+\zeta\left(t-2\delta+\frac{(2n-1)}{n}\delta\right)-\zeta\left(t-2\delta+\frac{(2n-2)}{n}\delta\right)+\zeta(t-2\delta+2\delta)-\zeta\left(t-2\delta+\frac{(2n-1)}{n}\delta\right)\right]$$

Writing out ⊛⊛ for i=1,...,n we obtain

$$\frac{\delta}{n}\left[\zeta\left(t-2\delta+\frac{(n+1)}{n}\delta\right)-\zeta\left(t-\delta+\frac{\delta}{n}\right)+...+\zeta(t-\delta+\delta)-\zeta\left(t-\delta+\frac{(n-1)}{n}\delta\right)+\zeta\left(t-\delta+\frac{(n+1)}{n}\delta\right)-\zeta(t-\delta+\delta)\right]$$

$$\vdots$$

$$\frac{\delta}{n}\left[\zeta\left(t-\delta+\frac{(n+1)}{n}\delta\right)-\zeta(t-\delta+\delta)+...+\left(t-\delta+\frac{(2n-1)}{n}\delta\right)-\zeta\left(t-\delta+\frac{(2n-2)}{n}\delta\right)+\zeta(t-\delta+2\delta)-\zeta\left(t-\delta+\frac{(2n-1)}{n}\delta\right)\right]$$

Multiplying ⊛, $i=1,\dots,n$ by ⊛⊛ for $i=1,\dots,n$ and taking expectations gives:

$$\left(\frac{\delta^2}{n^2}(n-1)\frac{\delta}{n} + \frac{\delta^2}{n^2}(n-2)\frac{\delta}{n} + \dots + \frac{\delta^2}{n^2}\Big(n-(n-1)\Big)\frac{\delta}{n}\right)\Omega$$ (by 3.4.1 and 3.4.2)

$$\vdots$$

$$\frac{\delta^2}{n^2}\left(n-(n-1)\frac{\delta}{n}\right)\Omega$$

$$0 \ .$$

Summing these terms gives:

$$\left(\frac{\delta^3}{n^3}(n-1) + \frac{\delta^3}{n^3}2(n-2) + \dots + \frac{\delta^3}{n^3}(n-1)\Big(n-(n-1)\Big)\right)\Omega$$

Therefore (3.5.3.1) reduces to:

$$\lim_{n\to\infty} \frac{\delta^3}{n} \sum_{m=1}^{n} \frac{m}{n}\left(1-\frac{m}{n}\right)\Omega$$

$$= \lim_{n\to\infty} \delta^3 \sum_{i=1}^{n} \frac{i}{n}\left(1-\frac{i}{n}\right)\left(\frac{i}{n} - \frac{(i-1)}{n}\right)\Omega$$

which is by definition the Stieltjies integral

$$\delta^3 \int_0^1 x(1-x)\,dx \ \Omega$$

$$= \frac{\delta^3}{6}\Omega \ .$$

Similarly the covariance matrix ϕ_0

$$= E\left(\int_{t-\delta}^{t}\int_{s-\delta}^{s} d\zeta(\theta)\,ds\right)\left(\int_{t-\delta}^{t}\int_{s-\delta}^{s} d\zeta(\theta)\,ds\right)'$$

$$= \frac{2}{3}\delta^3\Omega$$

It follows from (3.4.2) that $\phi_k = 0, \ \forall k > 1$ and we know that

$\phi_k = \phi_{-k}, \ \forall k$.

Therefore the autocovariance function of ξ_t

$$H(L) = \delta^3\left(\frac{1}{6}L + \frac{1}{6}L^{-1} + \frac{2}{3}\right)\Omega$$

$$= \delta^3(1 + 0{,}268L)(1 + 0{,}268L^{-1})\Omega \quad , \qquad (3.5.3.4)$$

Kendall and Stuart Vol.3 (1966) show that if $H(L)$ factors into

$G(L)G'(L^{-1})C(L)$ then,

$$\xi_t = G(L)\varepsilon_t$$

where ε_t has covariance generating function $C(L)$.

(3.5.3.2) is of the above form with $C(L)$ a constant.

Therefore

$$G(L) = 1 + 0{,}268L$$

and

$$\xi_t = (1 + 0{,}268L)\varepsilon_t \qquad (3.5.3.5)$$

where ε_t is serially uncorrelated.

The system (3.5.3.1) may therefore be transformed into one which has serially uncorrelated error terms by multiplying by the inverse of the above process.

$$\varepsilon_t = (1 + 0{,}268L)^{-1}\xi_t.$$

Expanding by Taylor series and truncating after 3 terms we obtain

$$\varepsilon_t = \xi_t - 0{,}268\xi_{t-1} - 0{,}268^2\xi_{t-2} - 0{,}268^3\xi_{t-3} \quad .$$

We transform variables of the form x_t^0 in (3.5.3.1) to $*x_t^0$ by

$$*x_t^0 = x_t^0 - 0{,}268x_{t-1}^0 + 0{,}268^2x_{t-2}^0 + 0{,}268^3x_{t-3}^0 \quad .$$

All variables being in the form $*x_t^0$ implies that the error term in (3.5.3.1) will be serially uncorrelated. It would therefore be possible to estimate the parameters of such a system using Full Information Maximum Likelihood (See Appendix A).

3.5.4 THE DISCRETE APPROXIMATION TO THE MODEL

If, in the stock or mixed equation systems (3.5.2.4) and (3.5.3.2) a priori restrictions are imposed on the A and B matrices then ψ_t (or ψ_t^0) becomes a complicated function of A. This makes estimation of these systems extremely expensive and an approximation has therefore been developed which does in fact preserve the error structure (see Wymer (1976)) This approximate model may be used as an alternative or at least to obtain initial estimates of the parameters which can then be used in the Exact system.

In the case of an r^{th} order stock model we integrate (3.5.1.1) over the interval $(t-\delta,t)$ and making the substitution

$$\frac{1}{\delta}\int_0^\delta dx(t-s) = \Delta x_t, \frac{1}{\delta}\int_0^\delta x(t-s)ds \hat{=} Mx_t$$

where $\Delta = \frac{1}{\delta}(1-L)$, $M = \frac{1}{2}(1+L)$; L is the lag operator

we obtain as a discrete analogue to (3.5.1.1):

$$\Delta^r y_t^* = \sum_{i=1}^{r} A_i^o(M^{r-i+1}\Delta^{i-1})y_t^* + M^r B^* z_t + v_t^* \quad (3.5.4.1)$$

As before this model can be written as the following first order system:

$$\Delta y_t = AMy_t + BMz_t + v_t$$

Recall (3.5.2.2) that:

$$w_t = \int_0^\delta e^{As} d\zeta(t-s)$$

Wymer has shown (1972) that:

$$v_t = \frac{1}{\delta}(I - \tfrac{1}{2}\delta A)w_t$$

so that:

$$\Omega_v = E(v_t v_t') = \frac{1}{\delta}(I - \tfrac{1}{2}\delta A)\left\{\int_0^\delta e^{sA}\Omega e^{sA'} ds\right\}\frac{1}{\delta}(I-\tfrac{1}{2}\delta A)$$

and $E(v_t v'_{t-k}) = 0, \forall k$

The v_t are thus contemporareously correlated only.

It may further be shown (Wymer(1972)) that:

$$\Omega_v = \Gamma_r \otimes \Omega$$

where

$$\Gamma_r = \begin{bmatrix} \gamma_{r-1} & 0 & \cdots & 0 \\ 0 & \gamma_{r-2} & & \\ \vdots & & \ddots & \\ 0 & & & \gamma_0 \end{bmatrix}$$

$$\gamma_r = \frac{\delta^{2r-1}}{(r!)^2(2r+1)(2r-1)}\{(2r-1) + \tfrac{1}{4}r^2(2r+1) - \tfrac{1}{2}(2r+1)(2r-1)\}.$$

The autocovariance generating function of v_t

$$= \sum_{k=0}^{\infty} \phi_k L^k$$

where $\phi_k = E(v_t v'_{t-k})$

$$= \Gamma_r \otimes \Omega \text{ (independent of } L\text{)}.$$

We know therefore that $v_t = B(L)\varepsilon_t$ where the ε_t are serially uncorrelated and

$$B(L)B'(L^{-1}) = \Gamma_r \otimes \Omega$$

which is independent of L.

Now we have that (Wymer (1972))

$$v_t^* = [p'(L) \otimes I_n] v_t$$

where $p(L)$ is a vector of length r whose i^{th} element is:

$$\left(\frac{1}{\delta}(1-L)^{r-i}(\tfrac{1}{2}(1+L)\right)^{i-1}.$$

Therefore,

$$[p'(L) \otimes I_n]^{-1} v_t^* = B(L)\varepsilon_t$$

- a mixed system of the form $Av_t^* = B\varepsilon_t$.

Following Kendall and Stuart (1966), Vol.3, p.511 we see that v_t^* will have autocovariance generating function

$$
\begin{aligned}
F(L) &= A^{-1}(L)B(L)B'(L^{-1})A'^{-1}(L^{-1}) \\
&= (p'(L)\otimes I_n)\Gamma_r\otimes\Omega\left(p(L^{-1})\otimes I_n\right) \\
&= p'(L)\Gamma_r p(L^{-1})\Omega .
\end{aligned}
$$

Then, for example if $r = 2$

$$
= \frac{1}{\delta}L^{-1}\left(-\frac{1}{12}(1-L)^2 + \tfrac{1}{4}(1+L)^2\right)\Omega
$$

which is proportional to:

$$
(1 + 0{,}268L)(1 + 0{,}268L^{-1})\Omega .
$$

Therefore for $r = 2$, $v_t^* = (1 + 0{,}268L)\varepsilon_t$ with ε_t serially uncorrelated.

3.6 CONCLUSIONS

We have thus derived the error structure of the mixed first order system (exact form) - the discretization does not alter the error structure (Wymer (1976), as well as a general form for the error structure of an r^{th} order system for a purely stock model.

For the mixed model differential equation system of order 1 we derived a moving average error term of order 1, for the r^{th} order stock model we derived a moving average of order r-1. However, since the stock and flow variables are only mathematically distinguishable by the fact that the latter is a rate of change of a stock variables, one may write flow variable as $x' = Dx$; x' a flow variable, x a stock variable. Then rewriting the system in terms of stock variables alone, one will transform an r^{th} order system to an r^{th} or $(r+1)^{th}$ order system depending on whether the flow variables were determined in the equation system up to order r or not. Following the reasoning of Hurwitz (1977) it follows by induction that a mixed order model of order r, redefinable as either a model of order r or r+1 (dependent on the conditions above) has a moving average error term of order r-1, or r respectively.

CHAPTER FOUR

SIMULTANEOUS DIFFERENTIAL EQUATION MODELS OF THE SOUTH AFRICAN ECONOMY

4.1 INTRODUCTION

Chapter 2 concerned itself with the development of a linear forecasting model. The formulation was, however, comparative-static and gave no indication of the future growth paths of the variables under consideration. In this model we consider the application of two cyclical growth models to the South African economy. The models with their accompanying theoretical background will be presented separately. In addition, the steady state solution will be derived for each model in turn, estimation methods and a discussion of parameter estimates will be given, and then a detailed analysis of the asymptotic stability of the model follows.

The essential purpose of this study lies in the interpretation of the estimated structural parameters, that is, elasticities, propensities and speed of adjustment coefficients. Their values will give insight into the sensitivity of various components of Gross Domestic Product (G.D.P.) to changes in the exogenous variables and the time lags involved in the adjustment process. Through feedback mechanisms these changes will affect the model simultaneously with varying time lags and hence determine new growth paths for the endogenous variables. It is thus possible to analyse, for example, the result of the effects of government expenditure policy on the business cycle.

4.2 FORMULATION OF THE STRUCTURAL EQUATIONS

The general form of the equations below is

$$Dy(t) = \gamma(\hat{y}(t) - y(t)) \quad ; \quad D = \frac{d}{dt}, \ \gamma \text{ constant}$$

where $\hat{y}(t)$ is the optimal or desired value of y at time t. $\hat{y}(t)$ will usually be a function of the other endogenous variables in the system.

The obvious interpretation of the above is that of "partial equilibrium adjustment", that is, the rate of change of y (towards equilibrium) is proportional to the difference between the desired and actual y.

Solving the above system we obtain

$$y(t) = \gamma \int_0^\infty e^{-\gamma r} \hat{y}(t - r) dr$$

or that $y(t)$ depends with a distributed time lag on $\hat{y}(t)$.

Now

$$E(r) = \gamma \int_0^\infty re^{-\gamma r} dr = \frac{1}{\gamma}$$

Therefore if we have initially $y(0) = \hat{y}(0)$ and then $\hat{y}$ increases by some amount and then remains constant, $y(t)$ will gradually adjust over a time interval of ∞ to $\hat{y}$; the mean of the times it takes for changes in y to occur will be $\frac{1}{\gamma}$.

In addition

$$\int_0^{\frac{1}{\gamma}} \gamma e^{-\gamma r} dr = 0,632$$

Therefore $\frac{1}{\gamma}$ can be interpreted as the time required for 63.2% of the change that will be brought about in $y(t)$ to have occurred. γ is referred to as a speed of adjustment parameter; $\frac{1}{\gamma}$ as the mean time-lag.

4.3 FEEDBACK MECHANISMS

The models being considered endogenize the private sector and in the latter model a foreign sector - the government sector is taken as exogenous in both models. The behavioural equations determine demand and supply functions for goods traded in these sectors. Constraints on these functions take the form of identities in the model. In the real sector it is usually assumed that inventories absorb the initial impact of any discrepancy between supply and demand of goods in that sector, or of changes in exogenous levels of demand. Equilibrium is then regained through various feed back mechanisms which effect other variables in the model. Consider for example the model presented below, (4.4). An increase in government expenditure will result initially in a decrease of inventories. This will then feedback to cause an increase in income and thus consumption, investment and imports and eventually a return to equilibrium.

4.4 MODEL 1 - PARTIAL EQUILIBRIUM ADJUSTMENT (LINEAR FORM)

4.4.1 Specification of the Linear Model

The model below is, apart from minor modifications, that of Bergstrom (1967). It is a highly aggregative model of an open sector economy comprising 4 first order behavioural equations, 2 zero order behavioural equations and one identity. Exports and government expenditures are taken as exogenous variables. Two sets of data were used to estimate the parameters; yearly data 1946-1975 and quarterly data 1960-1974. In the first case data was at 1963 constant prices and in the second at 1970 constant prices.

When estimating the model for yearly data, dummy variables were included for the period 1960-1962 to adjust for the dis-equilibrating effects of Sharpeville and its aftermath.

The equations of the model estimated with annual data are given below. For estimation with quarterly data the dummy variables were removed.

1) $DC = \alpha(\hat{C} - C)$

$\hat{C} = (1-s)(Y-T) + d_1 D + A$

2) $DK = \gamma(\hat{K}-K)$

$\hat{K} = \nu Y + d_2 D$

3) $DY = \lambda(\hat{Y}-Y) + \mu(\hat{S}-S)$

$\hat{Y} = C + DK + G + Exp - Imp$

$\hat{S} = b(C + DK + G + Exp) + c$

4) $Imp = m(C + DK + G + Exp) + d_3 D$

5) $T = \tau Y - B$

6) $DS = Y + Imp - Exp - G - C - DK$

(^ denotes a desired level; $D = \frac{d}{dt}$)

Y = real net income or output

C = real private consumption

K = private fixed capital stock

S = real inventories

Imp= real imports

T = real taxes minus real government transfers

Exp= real exports

G = real government expenditure on goods and services

D = 1 ; 1960-1962

0 ; otherwise

$s, \nu, b, c, m, \alpha, \gamma, \lambda, \tau, \mu, A, B$ = positive constants ($s, m, \tau, < 1$)

d_1, d_2, d_3, d_4 = negative constants.

4.4.2. Interpretation of the Equations (Model I)

1. Equation 1 is a representation of a consumption function in which consumption is assumed to adjust with a weighted lag (rather than instantaneously) to disposable income. It is assumed that corresponding to any given level of disposal income, Y-T there is a partial equilibrium level of consumption $(1-s)(Y-T) + A$ and that at any point in time the rate of change in consumption is proportional to the difference between the partial equilibrium (desired) consumption and actual consumption. (The parameter $\frac{1}{\alpha}$ is interpreted as the mean time lag in the adjustment of desired to actual consumption.) 'A' denotes the level of autonomous desired expenditure, that is, the consumption flow that is independent of the level of income; we assume in fact that consumption flow will be A at zero income. (1-s) represents the marginal propensity to consume - s being the marginal propensity to save, the assumptions being that for every unit increase in disposable income, consumption flow increases by 1-s units and that disposable income is either saved or spent on consumption. We have seen above (4.2) that the differential equation may be alternatively specified as

$$c(t) = (1-s)\alpha \int_0^\infty e^{-\alpha r}(Y-T)(t-r)dr + A$$

This implies that consumption is now a function of lagged disposable income (with exponentially decreasing weights) at all previous points in time. One interpretation (Friedman (1957)) is that consumption depends on the present value of expected future income, that is, total wealth in a Friedmanite sense, and that expectations concerning future income are based on past income. The dummy variable was included to adjust for the fact that desired consumption fell during the period 1960-1962 due to the political uncertainty and expectations of an unstable future. The fall in desired consumption could not under those conditions be explained by stagnating Disposable Income alone.

2. The interpretation of equation 2 runs along similar lines to that of equation 1. Firstly, we can take the obvious interpretation that corresponding to each level of income Y, or output, there is some desired or optimal level of fixed capital stock $vY(\hat{K})$. Investment in fixed capital (net investment) is proportional to the discrepancy between the desired and actual capital stocks. The classical interpretation of an optimal capital stock is that at which the ratio of the marginal productivity of capital to the marginal productivity of labour equals the ratio of the wage rate to the interest rate. Our optimal production function here, however, is

$$\hat{K} = vY$$

that is, optimal capital stock is independent of labour. This assumption does not usually bear up empirically. In the South African case, however, data series for the period (1946-1972) have shown the capital output ratio to be a constant with an

$R^2 = 0.994$ (de Jager (73)). Neglect of the labour force is possible in this case, presumably because of the availability of black labour and the absence of black trade unions.

Alternatively we may write the equation as

$$DK = \int_0^{\infty} \gamma v e^{-\gamma r} DY(t-r)dr$$

That is, net investment depends on the change in income with an exponentially decreasing distributed time lag. This theory is a development of the accelerator mechanism (see chapter 2) of induced investment.

The introduction of the dummy variable in this equation is especially relevant because the immediate post-Sharpeville era was characterized by a large outflow on the capital account, withdrawal of foreign loan facilities and lack of confidence in the future. This resulted in a severe cutback in investment in capital stock and the liquidation of a number of companies - the dummy variable compensates for these effects on the level of capital stock. The parameter $\frac{1}{\gamma}$ is interpreted as the mean time lag in the adjustment of desired to actual stock.

3. The equation describing the change in income rests on the idea that the rate of increase in output depends on the excess demand for home produced goods over output, and on excess demand for inventory stock over actual inventory stock. It is assumed that desired inventory stock is a function of gross domestic expenditure (sales); increased economic activity will be reflected in higher desired stock. The parameter b will represent

the partial equilibrium ratio of stocks to sales. $\frac{1}{\lambda}$ can be interpreted as the mean time lag for the adjustment of output to sales; $\frac{1}{\mu}$ as the mean time lag in the adjustment of stocks to sales.

4. Imports are assumed to be a linear function of gross domestic expenditure; that is, we assume that a fixed proportion of sales are those of imported goods. Because of her reliance on imported capital goods in the mining and manufacturing sectors, South Africa has a relatively inflexible marginal propensity to import (parameter m)- this formulation is thus realistic. As before we incorporate a dummy variable to reflect the slowdown in imported goods as the government attempted to correct the serious Balance of Payment deficit on Capital Account in the post-Sharpeville period.

5. The specification of Personal Tax implies that tax is an increasing linear function of tax rates and that the proportion $\frac{T}{Y}$ increase as Y increases, that is a system of progressive tax rates (see chapter 2 for a fuller discussion). The dummy variable reflects government policy of the period in question, to stimulate growth through fiscal measures.

6. This equation is the national income identity, that is, that sales minus output equals change in inventories.

4.4.3. The Steady State Solution (Model I)

The equilibrium values of the endogenous variables denoted by (*) are as follows:-

$$C^* = \frac{(1-m)(1-s)(1-\tau)(Exp+G)+A+d_1D+(1-s)(B+d_4D)+(1-s)(1-\tau)(d_3D)}{1-(1-m)(1-s)(1-\tau)}$$

$$K^* = \nu Y^*$$

$$Y^* = \frac{(1-m)(Exp + G + A + d_1D) + (1-m)(1-s)(B + d_4D) + d_3D}{1-(1-m)(1-s)(1-\tau)}$$

$$Imp^* = \frac{m(Exp + G + A + d_1D + (1-s)(B + d_4D))}{1-(1-m)(1-s)(1-\tau)}$$

$$T^* = \tau Y^* - B$$

$$S^* = \frac{b(Exp + G + A + d_1D + (1-s)(B + d_4D))}{1-(1-m)(1-s)(1-\tau)} + c$$

Equilibrium income is affected by changes in Exp and G , the exogenous variables, or injections into the circular flow in the Keynesian sense. The income multiplier for changes in Exp or G is

$$\frac{(1-m)}{1-(1-m)(1-s)(1-\tau)}$$

that is the smaller the marginal propensities to save import and tax (the withdrawals from the system) the greater the effect of exogenous changes on equilibrium income. In addition the equilibrium value of income is dependent on B; a larger B is associated with lower tax flow or withdrawals at each level of income and hence a higher level of equilibrium income. Similar interpretations can be given for all the equilibrium levels of the real variables.

4.4.4. Time path of the endogenous variables (Model I)

Of particular interest is the deviation of the variables from their steady states especially when considering the stability of the system (see below). After elimination of I and T the system may be written as

$$Dy = Ay \qquad (4.4.4.1)$$

where

$$y = \begin{pmatrix} C-C^* \\ K-K^* \\ Y-Y^* \\ S-S^* \end{pmatrix}$$

and

$$A = \begin{vmatrix} -\alpha & 0 & \alpha(1-s)(1-\tau) & 0 \\ 0 & -\gamma & \gamma\nu & 0 \\ \lambda(1-m)+\mu b & -\gamma\lambda(1-m)-\mu b\gamma & \nu\gamma\lambda(1-m)-\lambda+\mu b\gamma\nu-\mu & \\ -(1-m) & \gamma(1-m) & 1-\gamma\nu(1-m) & a \end{vmatrix}$$

Bergstrom (1967).

We need the following theorem for our analysis

THEOREM 1

If A is non-singular and has distinct roots there exists an orthogonal P (matrix of its eigenvectors) such that

$$PAP^{-1} = \Lambda$$

where Λ is the diagonal matrix of the (distinct) roots of A (Bellman - (1970)).

We may write $Dy = Ay$ as

$$D(Py) = PAP^{-1}(Py)$$

$$\therefore D(Py) = \Lambda(Py)$$

$$\therefore Py(t) = \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & e^{\lambda_4 t} \end{bmatrix} Py(0)$$

$$\therefore y(t) = P^{-1}\begin{bmatrix} e^{\lambda_1 t} & \dots & \\ \vdots & \ddots & \\ \dots & \dots & e^{\lambda_4 t} \end{bmatrix} Py(0)$$

is a particular solution of (4.2.2.1). We can therefore express the time paths of the variables in terms of their initial values. (See also Appendix A for further clarification).

4.4.5. Estimation Procedures (Model I)

As expounded in chapter 3 parameters of linear models of the form

$$Dy(t) = A(\theta)y(t) + B(\theta)z(t) + u(t)$$

can be estimated by methods of Full Information Likelihood (F.I.M.L.). Because the model is not defined in terms of measurable quantities it is necessary to integrate it over the observation period; for example, in the case of a flow variable such as income

$$Y_t^o = \int_{t-1}^{t} Y(s)ds$$

is measurable as the flow of income over a period of one time interval (1 year or 1 quarter as the case may be) whereas the rate of flow at time t , $Y(t)$ is clearly not measurable. In addition, of course, stock variables which are observable will have to be transformed in the following way; for example, in the case of inventory level

$$S_t^o = \int_{t-1}^{t} S(s)ds \approx MS_t \text{ (trapezium rule)}$$

where $\quad M = \frac{1}{2}(1 + L) \qquad LS_t = S_{t-1}$

Therefore inventory level $S(t)$ which is observable must be transformed to $S_t^o = MS_t$ to be compatible with the above transformation of flow variables.

Since flow variables are determined in equations of order 1 in our model (for example Y) double integration will yield a model with a moving average error process of order 1 (see chapter 3).

In order to apply F.I.M.L. estimation techniques the variables must be prewhitened by the inverse of the moving average process; the error term will then be serially uncorrelated. The moving average process of order 1 is of the form (see Chapter 3)

$$\alpha(L) \quad = \quad 1 + 0{,}268L$$

The inverse of this process is thus (truncated to 4 terms)

$$\alpha^{-1}(L) \quad = \quad 1 - 0{,}268L + 0{,}268L^2 - 0{,}268^3L^3$$

Transformed variables will therefore be of the form $y^o_*(t)$ where

$$y^o_*(t) \quad = \quad \alpha^{-1}(L)\; y^o(t)$$

$y^o(t)$ equals the observable flow over the observation period δ in the case of $y(t)$ a flow variable or, $My(t)$ in the case of $y(t)$ an observable stock variable.

The approximate discrete form of the model is given below

$$\Delta \quad = \quad (1-L) \qquad M = \tfrac{1}{2}(1+L)$$

1. $\Delta C^o - \alpha\beta MY^o + \alpha\beta MT^o - 0{,}785\ \alpha d_1D^o - 0{,}785\ \alpha A + \alpha MC^o = 0$

2. $\Delta K^o - \gamma\nu MY^o - 0{,}785\ \gamma d_2D^o + \gamma MK^o = 0$

3. $\Delta Y^o - \lambda MC^o - \lambda\Delta K^o - \lambda MG^o - \lambda MExp^o + \lambda MY^o + \lambda MImp^o - \mu bMC^o$

   $- \mu b\Delta K^o - \mu bMG^o - \mu bMExp^o - 0{,}785\mu c + \mu MS^o = 0$

4. $MImp^o - mMC^o - m\Delta K^o - mMG^o - mMExp^o - 0{,}785\ d_3D^o = 0$

5. $MT^{O} - \gamma MY^{O} + 0{,}785\ B - 0{,}785\ d_4 D^{O} \quad = \quad 0$

6. $\Delta S^{O} - MY^{O} - MImp^{O} + MC^{O} + \Delta K^{O} + MExp^{O} + MC^{O} = 0$

Estimation by the program RESIMUL (Appendix C) requires the model to be written in the form

$$(I - \tfrac{1}{2}A)\Delta y \quad - \quad A\ Ly - BMz \quad = \quad 0$$

We obtain this form by replacing My by $(\tfrac{1}{2}\Delta + L)y$ for those endogenous variables determined in first order equations (Wymer 1975). For example, in the system above the substitution will be carried out for the endogenous variables C , K and Y but not for I and T .

We obtain the following (presentation as in Wymer (1975))

1. $f_1x_1 + \theta_1x_7 - f_2x_{14} - f_3x_2 - f_4x_8 + f_4x_5 - f_5x_{13} \quad = 0$

2. $f_6x_3 + \theta_3x_9 - f_7x_2 - f_8x_8 - f_9x_{13} \quad = 0$

3. $f_{10}x_2 - f_{11}x_1 - f_{12}x_3 + f_{14}x_6 - f_{13}x_7$

   $+\theta_5x_8 - f_{12}x_{11} - f_{12}x_{12} - f_{16}x_{14} + \theta_6x_{10} \quad = 0$

4. $c_1x_4 - f_{13}x_1 - \theta_8x_3 - \theta_8x_7 - \theta_8x_{11} - \theta_8x_{12} - \theta_{15}x_{13} = 0$

5. $c_1x_5 - f_{18}x_2 - \theta_9x_8 - \theta_{16}x_{13} + \theta_{12}x_{14} \quad = 0$

6. $c_1x_6 - c_2x_2 - c_1x_8 - c_1x_4 + c_1x_{12} + c_1x_{11} + c_2x_1$

   $+ c_1x_7 + c_1x_3 \quad = 0$

The variables, parameters and functions are given in the following table :-

Continuous Variables	Discrete Variables	Variables for model estimated by Resimul	Parameters	Functions
Endogeneous			$\theta_1 = \alpha$	$f_1 = c_1+c_2\theta_1$
			$\theta_2 = 1-s$	$f_2 = \theta_1\theta_{10}$
DC^o	ΔC^o	$x_1 = \Delta C^o$	$\theta_3 = \gamma$	$f_3 = c_2\theta_1\theta_2$
DK^o	ΔK^o	$x_2 = \Delta K^o$	$\theta_4 = \nu$	$f_4 = \theta_1\theta_2$
DY^o	ΔY^o	$x_3 = \Delta Y^o$	$\theta_5 = \lambda$	$f_5 = \theta_1\theta_{13}$
Imp^o	$M\ Imp^o$	$x_4 = M\ Imp^o$	$\theta_6 = \mu$	$f_6 = c_1+c_2\theta_3$
T^o	MT^o	$x_5 = MT^o$	$\theta_7 = b$	$f_7 = c_2\theta_3\theta_4$
DS^o	ΔS^o	$x_6 = \Delta S^o$	$\theta_8 = m$	$f_8 = \theta_3\theta_4$
			$\theta_9 = \tau$	$f_9 = \theta_3\theta_{14}$
Predetermined			$\theta_{10} = A$	$f_{10} = c_1+c_2\theta_5$
			$\theta_{11} = c$	$f_{11} = c_2\theta_5+c_2\theta_6\theta_7$
C^o	MC^o	$x_7 = LC^o$	$\theta_{12} = B$	$f_{12} = \theta_5+\theta_6\theta_7$
Y^o	MY^o	$x_8 = LY^o$	$\theta_{13} = d_1$	$f_{13} = \theta_5+c_2\theta_6\theta_7$
K^o	MK^o	$x_9 = LK^o$	$\theta_{14} = d_2$	$f_{14} = c_2\theta_6$
S^o	MS^o	$x_{10} = LS^o$	$\theta_{15} = d_3$	$f_{15} = \theta_6\theta_{15}$
			$\theta_{16} = d_4$	$f_{16} = \theta_6\theta_{11}$
Exogeneous				$f_{17} = c_2\theta_8$
Exp^o	$M\ Exp^o$	$x_{11} = M\ Exp^o$		$f_{18} = c_2\theta_9$
G^o	MG^o	$x_{12} = MG^o$		
D^o	$0{,}785\ D^o$	$x_{13} = 0{,}785\ D^o$		
1,0	0,785	$x_{14} = 0{,}785$		

$$c_1 = 1{,}0$$

$$c_2 = 0{,}5$$

4.4.6. Parameter Estimates (Model I)

Estimates of the θ_i were obtained using the computer program TRANSF and RESIMUL available on the UNIVAC 1106 computer, and written by C.R. Wymer of the London School of Economics. (TRANSF is used to produce a data matrix on disc for input to RESIMUL which uses Full-Information Maximum Likelihood estimation procedures to calculate the θ_i. Appendix A).

The FIML estimates are found by a Newton-Raphson iterative procedure with arbitrary initial values of the parameters (see Appendix A). Convergence is assumed and the iterative procedure stopped when the maximum proportional change in the parameters is less than some fixed quantity ε usually set at 0,01 or 0,001, or when the proportional change in the log likelihood function is less than 10^{-6}.

Using a criterion of $\varepsilon = 0{,}01$ the following parameter values were obtained.

Parameter	Estimate	t-statistic
α	1,41675	7,49
1-s	0,70832	63,74
γ	0,12141	3,16
ν	3,77644	10,65
λ	1,90363	3,42
μ	0,31759	2,47
b	0,80751	4,37
m	0,23922	45,12

Parameter	Estimates	t-statistic
A	450.28096	7,17
τ	0,08433	9,20
c	-425,38138	2,77
B	92.30049	1,58
d_1	-184,28963	2,52
d_2	-2400,63215	2,23
d_3	-164,03813	2,53
d_4	-65,55342	1,18

The 5% significance level of a one-sided t-statistic with 30 degrees of freedom is 1,70

However, these parameter estimates gave rise to the following error message;

"Pivot 16 of the matrix being invented was diagonal element 14 whose value = $0{,}8667 \times 10^{-7}$."

This infers that the inversion of the Hessian matrix calculated in the Newton-Raphson iterative procedure has given rise to small pivots. As Wymer (1975) states ...

"It is emphasised that a set of estimates should not be accepted as maximum likelihood estimates if this message appears in the last few iterations of the estimation procedure."

Parameter 14 was thus replaced by its value at this last iteration namely -2400,63 .

The following estimates were then obtained,

Table 4.1 for a criterion of ε = 0,01 and

Table 4.2 for a criterion of ε = 0,001 (with no error Messages in both cases).

Published work on FIML estimation (Sheen (1976), Jonson (1977) Wymer (1974) has been confined to models with variables in log form and thus with small variances. The variables in this model exhibited large variances and convergence was slow and involved large fluctuations in the parameter values (see below). The work on log models, (see 4.5) yielded parameter estimates which converged monotonically with small parameter fluctuations.

	Table 4.1 (ε = 0,01)		Table 4.2 (ε = 0,001)	
	Estimates	t value	Estimates	t value
α	1,25643	6,97	1,14644	6,29
1-s	0,71121	62,80	0,71291	60,19
γ	0,10549	4,32	0,08624	3,75
ν	4,00495	12,52	4,27234	9,43
λ	1,87029	3,50	2,08493	4,00
μ	0,25946	1,99	0,02188	0,16
b	0,85827	3,68	4,73384	0,18
m	0,23981	45,75	0,24110	45,70
τ	0,08478	9,34	0.08604	9,50
A	451,72095	7,08	455,71556	6,88
c	-375,30259	2,48	-361,92171	1,85
B	92,02591	1,58	102,85480	1,78
d_1	-183,22315	2,61	-174,04301	2,46
d_3	-156,56246	2,61	-155,80495	2,57
d_4	-58,80168	1,16	-43,52036	0,87

Using a criterion of ε = 0,01 thirteen out of fifteen parameters were significant at the 5% level. At ε = 0,001 twelve parameters were significant.

4.4.7. Interpretation of the parameter estimates (Model I) (Table 4.2)

The speed of adjustment coefficient for the consumption equation indicates that actual consumption adjusts to desired consumption with a mean time lag of ten months.

The values of (1-s), the marginal propensity to consume and A the level of autonomous consumption are realistic. They infer that any 1 rand increase in disposable income results in an increase in desired consumption of approximately 71 cents, and that consumption which is quite independent of income amounts to 455 million rand.

The value of d_1 indicates that during the period 1960-1962 desired consumption was 174 million rand lower than one would expect given disposable income and fixed autonomous consumption.

The value of γ indicates a long mean time lag of desired to actual capital stock of 12 years. The desired capital to output ratio is estimated as 4,27. The value of d_2 is indicative of the sharp fall in desired capital stock in the post Sharpeville period; a shortfall of some 2400 million rand below that expected, given the capital output ratio and the income flow.

The value of λ indicates that the mean time lag in adjustment from output to sales is about 6 months.

μ has a similar interpretation but the non-significant t-statistic of μ and b indicate that the entrepreneurs are not looking at stock levels when considering output adjustment.

Theorem 2

Let $Dy(t) = Ay + f(t,y)$

where A is a real constant matrix with the characteristic roots all having negative real parts. If f is real and continuous for small $|y|$ and $t > 0$; and if

$$\frac{f(t,y)}{|y|}$$

tends to zero uniformly in t as $|y|$ tends to zero, then the identically zero solution is asymtotically stable.

$f(t,y)$ usually occurs as the residual in a Taylor series expansion of non-linear expressions, see for example Wymer (1974) As shown above (4.4.4.1), however, the model can be written as $Dy = Ay$ and hence the question of uniform convergence of $f(y,t)$ does not arise, with

$$y = \begin{bmatrix} C-C^* \\ K-K^* \\ Y-Y^* \\ S-S^* \end{bmatrix}$$

An asymptotically stable solution of the form $y(t) = 0$ will exist and the vector

$$\begin{bmatrix} C \\ K \\ Y \\ S \end{bmatrix}$$

will converge to its steady state

$$\begin{bmatrix} C^* \\ K^* \\ Y^* \\ S^* \end{bmatrix}$$

as t tends to infinity, if A has eigenvalues with negative real parts (see Appendix A for a note on the solution of differential equations of the form $Dy = Ay$).

The characteristic roots of A are the roots of

$$x^4 + a_1x^3 + a_2 x^2 + a_3x + a_4 = 0$$

where

$$a_1 = \alpha + \gamma + \lambda - \gamma\lambda\nu(1-m) - \mu b\gamma\nu$$

$$a_2 = \alpha\gamma + \gamma\lambda + \alpha\lambda \{1 - (1-m)(1-s)(1-\tau)\}$$
$$-\alpha\gamma\lambda\nu(1-m) + \mu\{1-\alpha b(1-s)(1-\tau) - \gamma\nu(1-m) - \alpha b\gamma\nu\}$$

$$a_3 = \alpha\gamma\lambda\{1 - (1-m)(1-s)(1-\tau)\}$$
$$+ \mu[\gamma+\alpha\{1-(1-m)(1-s)(1-\tau)\}-\alpha\gamma b(1-s)(1-\tau)-\alpha\gamma\nu(1-m)]$$

$$a_4 = \mu\alpha\gamma\{1 - (1-m)(1-s)(1-\tau)\}$$

Necessary and sufficient conditions for the roots of the above equation to have negative real parts are given by the Routh-Hurwitz conditions (Samuelson (1947)). Before we state the theorem we construct the following square matrix. For any given polynomial of the n^{th} degree with coefficients a_i ; $i=0,1,\ldots,n$ list the odd coefficients in a row, treating all coefficients

Therefore :

$$\Delta_1 = 2{,}69873 > 0$$
$$\Delta_2 = 1{,}82471 > 0$$
$$\Delta_3 = 0{,}18365 > 0$$
$$\Delta_4 = 0{,}00109 > 0$$

The system therefore has an asymptotically stable solution.

4.5 MODEL II - PARTIAL EQUILIBRIUM (LOG-LINEAR FORM)

The stimulus for constructing a log-linear model stems from two factors.

Firstly, the estimation of the linear system of differential equations raises one important theoretical point. Consider, for example the consumption function which makes desired consumption flow a linear function of disposable income. This infers that the desired income elasticity of consumption approaches unity for large disposable income, an unrealistic assimption (see chapter 2). A more realistic aggregative assumption is that the disposable income elasticity of consumption is constant ($\neq 1$). Expressing desired or optimal flows in terms of constant elasticity variables requires for estimation purposes that the model be expressed in terms of log-linear variables.

Secondly, a log-linear system of the South African monetary sector has been developed by Hurwitz and Kantor (1977). The development of a real sector log-linear system would then allow the integration of the two models to form a model representing the functioning of the economy as a whole.

4.5.1 A Note on Elasticity

For simplicity we consider the case of one independent variable. Given $y = f(x)$ the elasticity of y with respect to x is usually thought of as the ratio of the percentage change in y to the percentage change in x (sometimes called the arc elasticity of y with respect to x)

OR $$\varepsilon_{yx} = \frac{\% \Delta y}{\% \Delta x} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y}$$

Given that f is continuous we may take the limit as Δx tends to zero and obtain what is known as the point elasticity of y with respect to x.

that is $$\varepsilon_{yx} = \frac{dy}{dx} \cdot \frac{x}{y}$$

OR $$\varepsilon_{yx} = \frac{d\,(\ln y)}{d\,(\ln x)}$$

Theorem 3

If $y = x^{\alpha}$; α, a constant, then the elasticity of y with respect to x is α.

Proof.

If $y = x^{\alpha}$

then $\ln y = \alpha \ln y$

$$\varepsilon_{yx} = \frac{d \ln y}{d \ln x} = \alpha$$

Why sudden proof now?

4.5.2. Specification of the Log-Linear Model

The model consists of 5 behavioural equations and 3 identities ...

1. $$D \text{ Log } C = \gamma_1(\text{Log } \hat{C} - \text{Log } C) + u_1$$

$$\hat{C} = C_o(D_s Y)^{\beta 1}$$

2. $$D \text{ Log } I = \gamma_2(\text{Log } \hat{I} - \text{Log } I) + u_2$$

$$\hat{I} = I_o \, e^{\beta_2(D \text{ Log } Y) - \beta_3(r - D \text{ Log } P)}$$

3. $$D \text{ Log Exp} = \gamma_3(\text{Log } \widehat{\text{Exp}} - \text{Log Exp}) + u_3$$

$$\widehat{\text{Exp}} = \text{Exp}_o \, e^{\lambda t} \left(\frac{P_{Exp}}{P}\right)^{\beta_4}$$

4. $$D \text{ Log Imp} = \gamma_4(\text{Log } \widehat{\text{Imp}} - \text{Log Imp}) + u_4$$

$$\widehat{\text{Imp}} = \text{Imp}_o \left(\frac{P_{Imp}}{P}\right)^{\beta_5} \text{Gex}$$

5. $$D \text{ Log } Y = \gamma_5(\text{Log } \hat{Y} - \text{Log } Y) + u_5$$

$$\hat{Y} = \left(1 - \text{Imp}_o \left(\frac{P_{Imp}}{P}\right)^{\beta_5}\right) \text{Gex}$$

6. $$D_s Y \equiv Y - T$$

7. $$\text{Gex} \equiv C + I + \text{Exp} + G$$

8. $$DS \equiv Y + \text{Imp} - C - I - \text{Exp} - G$$

$u_i; i=1,\ldots,5$ are disturbance terms.

All real variables are at 1970 prices.

C_o , I_o , Exp_o , Imp_o are constants.

Endogenous Variables

C = real consumption.

I = real investment.

Exp = real exports.

Imp = real imports.

Y = real income.

D_sY* = real disposable income.

Gex = real gross expenditure

S = real inventory level.

Exogenous Variables

T = real personal taxes minus real transfers.

G = real government expenditure.

P = wholesale price index.

P_{Imp} = import price index.

P_{Exp} = export price index.

r = short term interest rate (Treasury Bill rate).

This model is in many ways analogous to the model presented in part 1. The economic interpretation will therefore be brief.

* Note that in this chapter D_sY is used to denote disposable income.

4.5.3. Interpretation of the Equations (Model II)

The consumption function expresses the idea that desired consumption is related to disposable income with constant elasticity β_1 (see previous note). Any discrepancy between the log of the ratio, desired to actual consumption is assumed to cause proportional change in consumption with a mean time lag of $\frac{1}{\gamma_1}$. (Similar interpretations of the γ_i are possible for all the equations). As in the previous model we may write the solution to the consumption function as follows :-

$$\text{Log } C = \gamma\beta_1 \int_0^\infty e^{-\gamma r} \text{ Log } D_s Y(t-r)dr$$

that is Log C is dependent on income at all previous points of time, with less weight being attached to income in the more remote past.

The log of desired investment is assumed to be explained by changes in income (modified accelerator theory) and by changes in the real rate of interest. The latter represents the opportunity cost of investment, or the cost of finance if expenditure on capital stock is out of borrowed funds. We would expect, a priori, the sign of β_3 to be positive indicating that investment expenditure is inversely proportional to finance charges. Accelerating real income is assumed to cause increases in investment expenditure as firms adjust for expected increased sales. This specification did not in fact turn out to be entirely satisfactory because of a fluctuating value of β_2. Using income flows and not changes in flows of income provided satisfactory results.

It was possible to consider the export equation from two different view points; either as dependent on desired world demand or desired local supply. Considering it from the point of view of supply we would firstly expect suppliers to adjust output depending on the level of world economic activity (or demand). This level is proxied by an exponential time trend. Secondly, as South Africa is a price taker on commodity markets and these comprise a large proportion of exports, supply to foreign markets would be dependent on the ratio of prices paid for exports on foreign markets, to local prices.

The components consumption, investment and export, along with government expenditure which is taken as exogeneous to the system make up aggregate demand. This demand has to be met by local production, imports or inventory change. The proportions of demand met by domestic production or imports will be determined by their relative prices. The constraint that desired output and imports equals the sum of the components of aggregate demand is incorporated into the model.

Desired imports are taken to be a function, therefore of relative prices (ratio of import prices to local prices) and total expenditure. By our constraints the demand for domestic production is taken as being residually determined through the demand for imports.

Changes in stock will equal actual supply minus actual demand.

The system is completed by defining disposable income as income net of tax and gross expenditure as the sum of the components of realized demand.

4.5.4. The Steady State Solution (Model II)

Knowledge of the stability of the system is important because an unstable system could imply a structural defect in the model and anyway would seriously effect its usefulness in forecasting or government policy analysis.

The basic differential equation system may be written :-

$$Dy(t) = F(y(t), z(t), \theta) + u(t) \qquad (4.5.4.1)$$

(Wymer (1976))

where $y(t)$ is a vector of endogenous variables.

$z(t)$ is a vector of exogenous variables.

$u(t)$ is a vector of white noise.

θ is the vector of model parameters to be estimated.

The steady state of the system is determined by assuming that all exogenous variables exhibit constant exponential growth,

that is $z(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_i(t) \\ \vdots \\ z_p(t) \end{bmatrix}$ is replaced

in (4.5.4.1) by

$$z_i(t) = z_i^* e^{\lambda_i t} \quad , \text{ for each } i$$

z_i^* a constant.

If a steady state exists the system will have a particular solution

$$y_i(t) \;=\; y_i^* \, e^{\rho_i t} \qquad , \quad \text{for all} \quad i$$

where $\rho_i \;=\; g_i(\, \lambda \,,\, \theta \,)$

$$y_i \;=\; G_i(z^* \,,\, \lambda \,,\, \theta)$$

We therefore write the steady state paths of the exogenous variables T , G , P , P_{Exp} and P_{Imp} as

$$\begin{aligned} T(t) &= T^* \, e^{\rho_T t} \\ G(t) &= G^* \, e^{\rho_G t} \\ P(t) &= P^* \, e^{\rho_P t} \\ P_{Exp}(t) &= P_{Exp}^* \, e^{\rho_{P_{Exp}} t} \\ P_{Imp}(t) &= P_{Imp}^* \, e^{\rho_{P_{Imp}} t} \end{aligned}$$

with for example ρ_G being the steady state growth rate of government expenditure.

These steady state solutions are substituted into the system and by equating coefficients we may obtain steady state exponential growth rates for the endogenous variables. Note, however, that as some equations are not in log form they have to be suitably log-linearised by using Taylor Series expansions about their sample means.

The relevant expansions are :-

$$\log(1-x) = \log(1-\exp \overline{\log x}) - \frac{(\log x - \overline{\log x})\exp \overline{\log x}}{1 - \exp \overline{\log x}}$$

$$\log(x \pm y) = \log(\exp \overline{\log x} \pm \exp \overline{\log y}) + \frac{\exp \overline{\log x}(\log x - \overline{\log x})}{\exp \overline{\log x} + \exp \overline{\log y}}$$

$$\pm \frac{\exp \overline{\log y}}{\exp \overline{\log x} + \exp \overline{\log y}} (\log y - \overline{\log y})$$

where bars denote sample means.

We obtain the following growth rates

Subscripts refer to the variable in question.

$$\rho_C = \beta_1 \; \rho_{D_s Y}$$

$$\rho_I = \beta_2 \; \rho_Y + \beta_3 \; \rho_P$$

$$\rho_{Exp} = \lambda_1 + \beta_4 \; \rho_{P_{Exp}} - \beta_4 \; \rho_P$$

$$\rho_{Imp} = \beta_5 \; \rho_{P_{Imp}} - \beta_5 \; \rho_P + \rho_{Gex}$$

$$\rho_Y = \rho_{Gex} - k_1 \; \beta_5 \; \rho_{P_{Imp}} + k_1 \; \beta_5 \; \rho_P$$

$$k_1 = \exp \overline{\log Imp_o \left(\frac{P_{Imp}}{P}\right)^{\beta_5}} \left\{1-\exp \overline{\log Imp_o \left(\frac{P_{Imp}}{P}\right)^{\beta_5}}\right\}^{-1}$$

$$\rho_{D_s Y} = k_2 \; \rho_Y - k_3 \; \rho_T$$

$$k_2 = \exp \overline{\log Y} \{\exp \overline{\log Y} + \exp \overline{\log T}\}^{-1}$$

$$k_3 = \exp \overline{\log T} \{\exp \overline{\log Y} + \exp \overline{\log T}\}^{-1}$$

$$\rho_{Gex} = k_4 \; \rho_C + k_5 \; \rho_G + k_6 \; \rho_{Exp} + k_7 \; \rho_I$$

Letting $k = \{ \exp \overline{\log C} + \exp \overline{\log G} + \exp \overline{\log Exp} + \exp \overline{\log Imp} \}^{-1}$

$$k_4 = k \exp \overline{\log C}$$

$$k_5 = k \exp \overline{\log G}$$

$$k_6 = k \exp \overline{\log Exp}$$

$$k_7 = k \exp \overline{\log Imp}$$

Our conclusions are what we might have expected. For example the growth of consumption expenditure is dependent on the growth of disposable income; the growth of imports is dependent positively on the growth of expenditure and has a negative correlation to the difference between the growth of foreign and local prices ($\beta_5 < 0$).

4.5.5 Estimation Procedures (Model II)

The procedures discussed in (4.4.3) are relevant to this section and presentation of the appropriate discrete model will be identical. Estimation did, however, require the log-linearization of the income equation. Log-linearizations have been found to be unsatisfactory (Hurwitz (1977)) producing large estimation error - thus the identities were excluded for estimation purposes.

The implication is that the Gex and D_sY are exogenous for estimation purposes but endogenous to the complete model.

The Taylor expansion linearization of the term

$$\log(1 - Imp_o(\frac{P_{Im}}{P})^{\beta_5})$$

in the income equation is given below.

$$\log\left(1 - \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}\right)$$

$$= \log\left(1 - \exp \overline{\log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}}\right)$$

$$\frac{-\left(\log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5} - \overline{\log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}}\right) \exp \overline{\log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}}}{1 - \exp \log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}}$$

The approximate discrete model is given below: -

1. $\Delta \log C^o + \gamma_1 M \log C^o - \gamma_1\beta_1 M \log D_s Y^o - \gamma_1 \log C_o$

2. $\Delta \log I^o + \gamma_2 M \log I^o - \gamma_2\beta_2 M \log Y^o + \gamma_2\beta_3 Mr^o$

   $-\gamma_2\beta_3 \Delta \log P^o - \gamma_2 \log I_o$

3. $\Delta \log \mathrm{Exp}^o + \gamma_3 M \log \mathrm{Exp}^o - \gamma_3\lambda_1(t-\bar{t}) - \gamma_3\beta_4 M \log P^o_{Exp}$

   $+\gamma_3\beta_4 M \log P^o - \gamma_3 \log \mathrm{Exp}_o$

4. $\Delta \log \mathrm{Imp}^o + \gamma_4 \log \mathrm{Imp}^o - \gamma_4\beta_5 M \log P^o_{Imp}$

   $+\gamma_4\beta_5 M \log P^o - \gamma_4 M \log \mathrm{Gex} - \gamma_4 \log \mathrm{Imp}_o$

5. $\Delta \log Y^o + k_1\gamma_5\beta_6 M \log P^o_{Imp} - k_1\gamma_5\beta_6 M \log P^o$

   $-\gamma_5 M \log \mathrm{Gex} + \gamma_5 M \log Y^o + k_1\gamma_5 \log \mathrm{Imp}_o - \gamma_5 k_2$

   where $k_1 = \dfrac{\bar{x}}{1-\bar{x}}$ $\quad k_2 = \dfrac{\bar{x}\log\bar{x}}{1-\bar{x}} + \log(1-\bar{x})$

   and $\bar{x} = \exp\left(\overline{\log \mathrm{Imp}_o\left(\frac{P_{Imp}}{P}\right)^{\beta_5}}\right)$

Writing the above in the form

$$(I - \tfrac{1}{2}A)\Delta y - ALy - BMz = 0$$

as before for RESIMUL we obtain

1. $f_1x_1 + \theta_1x_6 - f_2x_{11} - f_3x_{19} = 0$

2. $f_4x_2 + \theta_2x_7 - f_5x_5 + f_6x_{17} - f_6x_{18} - f_7x_{19} = 0$

3. $f_8x_3 - f_9x_{13} - f_{10}x_{16} + f_{10}x_{14} + \theta_3x_8 - f_{11}x_{19} = 0$

4. $f_{12}x_4 - f_{13}x_{15} + f_{13}x_{14} - \theta_4x_{12} + \theta_4x_9 - f_{14}x_{19} = 0$

5. $f_{15}x_5 + \theta_5x_{10} + f_{16}x_{15} - f_{16}x_{14} - \theta_5x_{16} - f_{17}x_{19} = 0$

The definitions of the f_i , x_j and θ_k are given in the following table.

Continuous Variables	Discrete Variables	Variables for Resimul	Parameters	Functions
Endogenous				
$D \log C^o$	$\Delta \log C^o$	$x_1 = \Delta \log C^o$	$\theta_1 = \gamma_1$	$f_1 = c_1 + c_2\theta_1$
$D \log I^o$	$\Delta \log I^o$	$x_2 = \Delta \log I^o$	$\theta_2 = \gamma_2$	$f_2 = \theta_1\theta_6$
$D \log Exp^o$	$\Delta \log Exp^o$	$x_3 = \Delta \log Exp^o$	$\theta_3 = \gamma_3$	$f_3 = \theta_1\theta_{12}$
$D \log Imp^o$	$\Delta \log Imp^o$	$x_4 = \Delta \log Imp^o$	$\theta_4 = \gamma_4$	$f_4 = c_1 + c_2\theta_2$
$D \log Y^o$	$\Delta \log Y^o$	$x_5 = \Delta \log Y^o$	$\theta_5 = \gamma_5$	$f_5 = \theta_2\theta_7$
			$\theta_6 = \beta_1$	$f_6 = \theta_2\theta_8$
			$\theta_7 = \beta_2$	$f_7 = \theta_2\theta_{13}$
Predetermined			$\theta_8 = \beta_3$	$f_8 = c_1 + c_2\theta_3$
			$\theta_9 = \beta_4$	$f_9 = \theta_3\theta_{11}$
$\log C^o$	$M \log C^o$	$x_6 = L \log C^o$	$\theta_{10} = \beta_5$	$f_{10} = \theta_3\theta_9$
$\log I^o$	$M \log I^o$	$x_7 = L \log I^o$	$\theta_{11} = \lambda_1$	$f_{11} = \theta_3\theta_{14}$
$\log Exp^o$	$M \log Exp^o$	$x_8 = L \log Exp^o$	$\theta_{12} = \log C_o$	$f_{12} = c_1 + c_2\theta_4$
$\log Imp^o$	$M \log Imp^o$	$x_9 = L \log Imp^o$	$\theta_{13} = \log I_o$	$f_{13} = \theta_4\theta_{10}$
$\log Y^o$	$M \log y^o$	$x_{10} = L \log y^o$	$\theta_{14} = \log Exp_o$	$f_{14} = \theta_4\theta_{15}$
			$\theta_{15} = \log Imp_o$	$f_{15} = c_1 + c_2\theta_5$
Exogenous				$f_{16} = c_3\theta_5\theta_{13}$
$\log D_sY^o$	$M \log D_sY^o$	$x_{11} = M \log D_sY$		$+ c_4\theta_5$
$\log Gex^o$	$M \log Gex^o$	$x_{12} = M \log Gex^o$		
$t - \bar{t}$	$0{,}785(t-\bar{t})$	$x_{13} = 0{,}785(t-\bar{t})$		
$\log P^o$	$M \log P^o$	$x_{14} = M \log P^o$		
$\log P^o_{Imp}$	$M \log P^o_{Imp}$	$x_{15} = M \log P^o_{Imp}$		
$\mathrm{Log}\ P^o_{Exp}$	$M \log P^o_{Exp}$	$x_{16} = M \log P^o_{Exp}$		
r^o	$M r^o$	$x_{17} = M r^o$		
$D \log P^o$	$\Delta \log P^o$	$x_{18} = \Delta \log P^o$		
1,0	0,785	$x_{19} = 0{,}785$		

$c_1 = 1{,}0$

$c_2 = 0{,}5$

$c_3 = 0{,}22$

$c_4 = -0{,}58$

4.5.6 Parameter Estimates (Model II)

Estimates were obtained using the programs TRANSF and RESIMUL as explained previously. The model was estimated using quarterly data for the period 1960 - 1974 and all variables at constant 1970 prices.

The convergence criterion used was $\varepsilon = 0{,}001$. The parameter estimates converged monotonically in a very small number of iterations with no error message being printed.

The estimates were :-

Parameter	Estimate	t-statistic
γ_1	0,30976	2,03
γ_2	0,20693	2,25
γ_3	2,63639	3,47
γ_4	0,38733	2,95
γ_5	0,63080	3,31
β_1	0,95580	15,94
β_2	1,56024	11,06
β_3	7,01158	2,04
β_4	0,69879	7,20
β_5	-0,48057	1,89
λ	0,00968	23,64
$\log C_o$	-0,04963	0,10
$\log I_o$	-5,73971	5,26
$\log Exp_o$	6,06165	284,89
$\log Imp_o$	-1,59329	46,06

The 5% (one-sided) significance point of the t-statistic with 60 degrees of freedom is 1,67 .

4.5.7. Interpretation of the Parameter Estimates (Model II)

All of the eleven parameters estimated were significant at the 5% level and three of the four estimated constants.

The signs of all the γ_i (speed of adjustment parameters) were positive as required. The value of $\gamma_1 = 0{,}3$ implies that the mean time lag for the adjustment of desired to actual log of consumption is approximately 10 months. Alternatively, the interpretation could be that a 1% increase in the log of desired over actual consumption will increase the rate of increase of consumption by approximately 0,3% per quarter; γ_2 to γ_5 have similar interpretations. The large value of γ_3 is interesting as it suggests that the supply of exports takes only one month on average (mean time lag) to reach the desired supply levels. A similar conclusion for the behaviour of exports was drawn by Hurwitz (1977).

The sign of β_1 , to β_5 were as expected. It is of interest that the relative price elasticity of desired supply of export was (0,69) , slightly higher in absolute value to the relative price elasticity of demand for imports (-0,48) . This infers that importers are less sensitive to price increases of imported goods than exporters are to price increases of exportable goods. The value of λ (0,009) implies that the real demand for South African export goods has stagnated over the time period under consideration, that is 1960 - 1974.

Can it be verified?

Comparison of the parameter estimates obtained for the two models is difficult because of their widely differing specifications. However, in the case where a comparison might be possible, notably in the consumption equation it is particularly encouraging to see that both models predict a speed of adjustment from desired to actual consumption of about 10 months.

4.5.8. The Stability Analysis (Model II)

The stability analysis was carried out using the computer program CONTINEST written by C.R. Wymer of the London School of Economics and edited for use on the University of Cape Town's Univac 1106 Computer by the author.

Given the approximate discrete model CONTINEST evaluates the reduced form; for a first order system we have

$$y_t = Ay_{t-1} + Bz_t$$

y_t , z_t have the normal interpretations.

The eigenvalues of the matrix A are then evaluated. According to our theorem (Perron) the system will be asymtotically stable if Bz_t converges uniformly with respect to t , and A has negative eigenvalues.

The eigenvalues of A were computed by CONTINEST and are given below:-

1 - 0,31059
2 - 0,20647
3 - 2,63565
4 - 0,38772
5 - 0,63214

(All imaginary parts zero).

The implication is, therefore, that under the assumption that Bz_t converges uniformly, the model is asymptotically stable. The assumption of uniform convergence for Bz_t is not very satisfactory; in order to avoid this complication, however, one would have to construct a completely endogenized model.

4.6 Conclusions - Incorporating monetary effects

The performance under estimation of model II was highly encouraging as all the parameter estimates of economic significance were realistic and significant. Sensitive as South Africa is to the political climate it does, however, seem necessary that one incorporate expectations in the form of Dummy variables, especially in the case of the Investment equation, to obtain a completely realistic picture of the economy. Standing on its own a real sector model of this nature has only limited value if the intention is to analyse the effect of government policy and price changes as no feedback mechanisms in money markets have been endogenized, - in other words the approach has been essentially Neo-Keynesian. By so doing we have ignored the Monetarist - Neo Keynesian debate which centres on the importance (or insignificance) of money on economic activity. In the next chapter we discuss this debate and analyse a model which considers the role of money as a determinant of national income.

4.7 The Direction of Further Research

As has been repeatedly stressed the logical step from this point is the incorporation of this log-linear real sector model with the monetary model of Hurwitz & Kantor (1977), thereby endogenizing the effects of money. Parameter estimation on such a model will begin shortly and it is to be hoped that such a model will be of value both from the point of view of giving further insight into the structure of the economy and providing accurate forecasts.

CHAPTER FIVE

MONETARISM VERSUS NEO-KEYNESIANISM - THE SOUTH AFRICAN CASE

5.1 INTRODUCTION

In the current controversy over the relative importance of fiscal and monetary measures in stabilization policy two conflicting schools of thought have evolved. The first group, known as Neo-Keynesians* believe that fiscal and monetary instruments are useful, but that the fiscal policy is the more potent tool of the two. The theory they propose is built on that developed in the General Theory of J.M. Keynes (1936) and later refined by J. Hicks (1937).

The second group de-emphasise the role of fiscal policy and regard monetary policy as the all important tool for economic regulation.

This debate has probably attracted more attention from a theoretical point of view than any other in stabilization policy analysis - the central theme of macro-economics. It might be stressed though that this chapter does not attempt to break any new theoretical ground. The stress is put on giving a

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*Neo-Keynesian theory for the purpose of this study is taken as the most prevelant line of thought amongst British economists who base their ideas on the General Theory. It therefore differs to some extent from the more sophisticated portfolio - risk theory of Tobin.

straightforward mathematical exposition of the economic theory, and in this way it is hoped that a reader unfamiliar with economics may grasp the essence of this argument without any considerable difficulty.

A mathematical treatment of the Keynesian and Monetarist views is presented first with a note on the mechanism of the causal influence of money on economic activity. With this theory as a background, measures of economic activity, fiscal, and monetary actions are selected and used to test the relative strength of fiscal and monetary actions on economic activity, using South African data. Therefore in addition, this study will give us some indication of the importance of monetary effects in a real sector analysis of the South African economy.

5.2 THE TWO VIEWPOINTS - AN IS-LM ANALYSIS

It should be pointed out from the beginning that most monetarists would consider an IS-LM interpretation of their views to be an oversimplification. They would argue that the IS-LM curves were not independent, with increases in money being reflected in shifts of both IS and LM curves due to the close relationship between money and demand. In addition, contemporary monetary theorists, following the work of Brunner and Meltzer (1974) would argue a sophisticated relative price - stock flow approach to the transmission mechanism (see below). For the purposes of this work, however, it provides the most useful paradigm from which to compare the two opposing views.

The analysis follows that of Hicks (1937) but has been extended to include the government sector.

We consider the following macro-economic model:

5.2.1 REAL SECTOR

$S = f(Y);\ \frac{\partial S}{\partial Y} > 0$

$T = f(Y);\ \frac{\partial T}{\partial Y} > 0$

$I = f(i_o);\ \frac{\partial I}{\partial i_o} < 0$

$S + T = G_o + I$ (Equilibrium condition)

ENDOGENOUS VARIABLES	EXOGENOUS VARIABLES
S - Savings	i_o - Interest rate
T - Tax receipts	G_o - Government expenditure
Y - National income	
I - Investment expenditure	

5.2.2 MONETARY SECTOR

$MD_1 = f(Y);\ \frac{\partial MD_1}{\partial Y} > 0$

$MD_2 = f(i_o);\ \frac{\partial MD_2}{\partial i_o} < 0$

$MD_1 + MD_2 = M_{so}$ (Equilibrium Condition)

ENDOGENOUS VARIABLES	EXOGENOUS VARIABLES
MD_1 = transactions demand for money	i_o = interest rate
MD_2 = speculative demand for money	M_{so} = Money supply

We have assumed therefore

(i) Savings and Tax are positively sloped functions of income, i.e. higher levels of income will be associated with higher levels of savings and taxes.

(ii) Investment expenditure is a negatively sloped function of interest rates. High rates of interest are associated with high opportunity costs of investment (and high finance charges) and Investment is assumed to be a positive function of the difference between the real rate of return on capital and real interest rates. Therefore any upward shift of interest rates will result in lower Investment flow.

(iii) Equilibrium is attained in the goods market where investments and government expenditure (injections to the circular flow of income) equal savings plus taxes (withdrawals from the circular flow of income). In the monetary sector equilibrium is attained where the demand and supply of money are equal.

(iv) Transactions demand for money is a positively sloped function of income because higher income (economic activity) will be associated with a higher demand for money for transaction purposes.

(v) The speculative demand for money is a negatively sloped function of the interest rate because high rates of interest will make bond holding relatively more attractive than holding speculative money balances.

5.2.3 THE IS CURVE

This is a plot of interest rate versus income for the goods market.

At equilibrium we have from 5.2.1

$$S(\bar{Y}) + T(\bar{Y}) - I(i_o) - G_o = 0$$

(where $\bar{Y}$ is equilibrium income).

Differentiating with respect to i_o we obtain

$$\frac{\partial S}{\partial \bar{Y}} \cdot \frac{\partial \bar{Y}}{\partial i_o} + \frac{\partial T}{\partial \bar{Y}} \cdot \frac{\partial \bar{Y}}{\partial i_o} - \frac{\partial I}{\partial i_o} = 0$$

$$\therefore \quad \frac{\partial \bar{Y}}{\partial i_o} = \frac{\frac{\partial I}{\partial i_o} \quad (< 0)}{\frac{\partial S}{\partial \bar{Y}} + \frac{\partial T}{\partial \bar{Y}} \quad (> 0)}$$

$$\therefore \quad \frac{\partial \bar{Y}}{\partial i_o} < 0$$

The economic explanation is that a decrease in the interest rate will result in an increase in Investment; in order to maintain the goods market identity income must rise, making savings and tax rise until

$$I + G_o = S + T$$

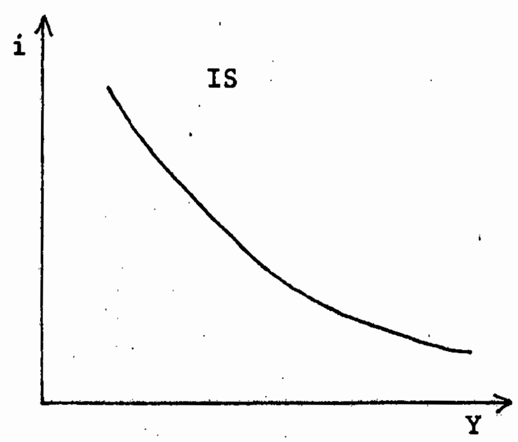


Figure 5.1 The IS Curve

5.2.4 THE LM CURVE

This is a plot of interest rate versus income for the money market.

At equilibrium we have

$$MD_1(\bar{Y}) + MD_2(i_o) = M_{s_o}$$

Differentiating with respect to i_o we obtain

$$\frac{\partial MD_1}{\partial \bar{Y}} \cdot \frac{\partial \bar{Y}}{\partial i_o} + \frac{\partial MD_2}{\partial i_o} = 0$$

$$\therefore \quad \frac{\partial \bar{Y}}{\partial i_o} = \frac{-\frac{\partial MD_2}{\partial i_o} (> 0)}{\frac{\partial MD_1}{\partial \bar{Y}} (> 0)}$$

$$\therefore \quad \frac{\partial \bar{Y}}{\partial i_o} > 0$$

The economic explanation is that increasing interest rates make money holding less attractive in comparison to bond holding. In order to make up the shortfall of money, income must increase the transactions component of monetary demand and make

$$MD_1 + MD_2 = M_{s_o}$$

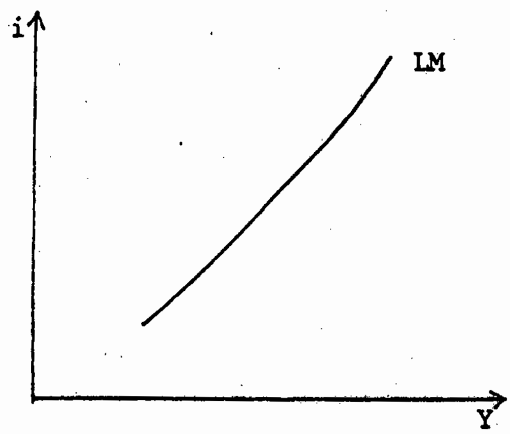


Figure 5.2 The LM Curve

Equilibrium in the goods and monetary markets is obtained by bringing the two analyses together, the interest rate will be endogenised and equilibrium will be attained at the intersection of the IS and LM curve. (See Figure 5.3)

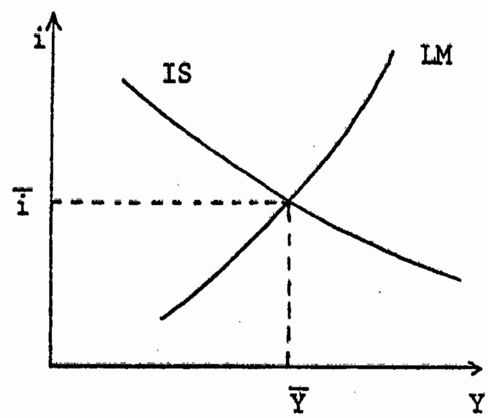


Figure 5.3 Equilibrium at the intersection of the IS and LM Curves

We now look at the effect on equilibrium income through variation in the exogenous variables G_o, M_{s_o}.

5.2.5 THE EFFECT OF FISCAL MEASURES

An exogenous increase in G_o will disturb equilibrium in the goods market; income must rise at each level of interest to regain equilibrium, i.e. the IS Curve will shift to the right. Equilibrium income and interest rate will rise. (See Figure 5.4.)

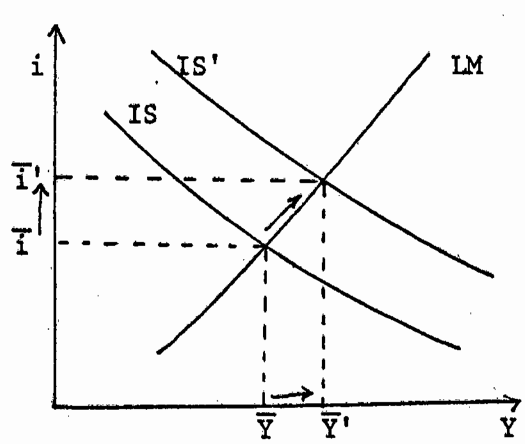


Figure 5.4 Outward Shift of the IS Curve

5.2.6 THE EFFECT OF MONETARY POLICY

Consider now an expansion of the monetary supply - at any set rate of interest and income level there now exists excess money. Equilibrium will be retained at any fixed interest rate by an increase in income, thereby increasing the transaction demand for money and making money demanded equal money supply. Therefore an increase in money produces an outward shift in the LM curve. Equilibrium income will rise and equilibrium interest rate will fall. (See Figure 5.5.)

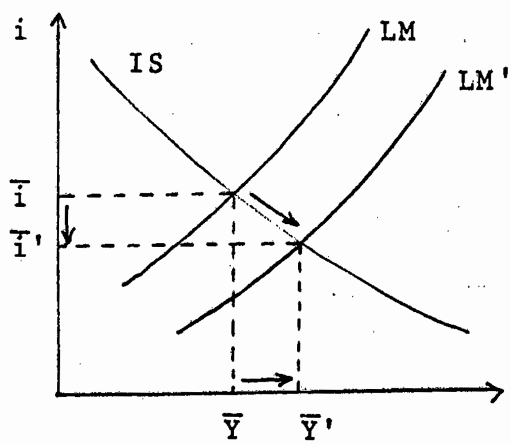


Figure 5.5 Outward Shift of the LM Curve

5.3 *A MATHEMATICAL TREATMENT OF THE EFFECT OF FISCAL AND MONETARY MEASURES ON EQUILIBRIUM INCOME

To quantify these effects exactly it is necessary to carry out a comparative static analysis. We make use of the implicit function theorem, Chiang (1974) for two endogenous variables, that is

Given the system

*Throughout this section we make use of the convention $\frac{\partial y}{\partial x} = y_x$.

$$F^1(y_1,y_2;\ x_1,x_2,\ldots,x_m)$$
$$F^2(y_1,y_2;\ x_1,x_2,\ldots,x_m) \qquad (5.3.1)$$

if

(a) the functions F^1 and F^2 have continuous partial derivatives with respect to all the y_i and x_i

and

(b) at a point $(y_{10},y_{20};\ x_{10},\ldots,x_{mo})$ satisfying (5.3.1) the Jacobian

$$|J| = \left|\frac{\partial(F^1,F^2)}{\partial(y_1,y_2)}\right| \quad \text{is non-zero}$$

then there exists an m dimensional neighbourhood of $(x_{10},\ldots,x_{mo})$, N, in which y_1 and y_2 are implicit functions of $x_1,\ldots,x_m$.

In addition these functions satisfy (5.3.1) for every m-tuple $(x_1,\ldots,x_m)$ in the neighbourhood N - thereby giving (5.3.1) the status of a set of identities in the neighbourhood N.

Our system is at equilibrium ((5.2.1) and (5.2.2)) when

$$S(Y) + T(Y) - I(i) - G_o = 0$$
$$MD_1(Y) + MD_2(i) - M_{s_o} = 0 \qquad (5.3.2)$$

We rewrite this system as

$$F^1(Y,i;\ G_o,M_{s_o}) = S(Y) + T(Y) - I(i) - G_o = 0$$
$$F^2(Y,i;\ G_o,M_{s_o}) = MD_1(Y) + MD_2(i) - M_{s_o} = 0$$

This system satisfies the implicit function theorem because

(i) we may assume F^1 and F^2 have continuous derivatives with respect to the variables in the model, since all the functions appearing in the model have been assumed to have continuous derivatives.

(ii) The endogenous variable Jacobian evaluated at the equilibrium equals

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial i} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial i} \end{vmatrix} = \begin{vmatrix} S_Y + T_Y & - I_i \\ MD_{1Y} & MD_{2i} \end{vmatrix}$$

$$= \underset{(>\ 0)}{(S_Y+T_Y)}\ \underset{(<\ 0)}{(MD_{2i})} + \underset{(<\ 0)}{I_i}\underset{(>\ 0)}{MD_{1Y}}$$

$$< 0$$

i.e. at equilibrium $|J| \neq 0$.

The implicit functions

$$\overline{Y} = \overline{Y}\ (G_o, M_{s_o})$$
$$\overline{i} = \overline{i}\ (G_o, M_{s_o})$$

therefore hold; in addition we may take (5.3.1) to be a pair of identities in a neighbourhood of the equilibrium and write

$$F^{1\prime}(\overline{Y},\overline{i};\ G_o,M_{s_o}) = S(\overline{Y}) + T(\overline{Y}) - I(\overline{i}) - G_o \equiv 0$$
$$F^{2\prime}(\overline{Y},\overline{i};\ G_o,M_{s_o}) = MD_1(\overline{Y}) + MD_2(\overline{i}) - M_{s_o} \equiv 0$$

We may therefore compute the following partial derivatives (Spiegel (1963) p 107)

$$\frac{\partial \overline{Y}}{\partial G_o} = - \frac{\begin{vmatrix} \frac{\partial F^{1\prime}}{\partial G_o} & \frac{\partial F^{1\prime}}{\partial \overline{i}} \\ \frac{\partial F^{2\prime}}{\partial G_o} & \frac{\partial F^{2\prime}}{\partial \overline{i}} \end{vmatrix}}{|J|} \qquad (5.3.3)$$

(J evaluated at $\overline{Y},\overline{i}$)

$$= \frac{MD_{2\bar{i}}}{MD_{2\bar{i}}\ (S_{\bar{Y}} + T_{\bar{Y}})\ +\ I_{\bar{i}}MD_{1\bar{Y}}} \quad \begin{matrix} < 0 \\ < 0 \end{matrix} \qquad (5.3.3)$$

Shorthand?

$$> 0$$

$$\frac{\partial \bar{Y}}{\partial M_{s_o}} = - \frac{\begin{vmatrix} \dfrac{\partial F^{1'}}{\partial M_{s_o}} & \dfrac{\partial F^{2'}}{\partial \bar{i}} \\ \dfrac{\partial F^{2'}}{\partial M_{s_o}} & \dfrac{\partial F^{2'}}{\partial \bar{i}} \end{vmatrix}}{|J|}$$

$$= \frac{I_{\bar{i}}}{MD_{2\bar{i}}\ (S_{\bar{Y}} + T_{\bar{Y}})\ +\ I_{\bar{i}}MD_{1\bar{Y}}} \quad \begin{matrix} < 0 \\ < 0 \end{matrix} \qquad (5.3.4)$$

$$> 0$$

5.4 THE NEO-KEYNESIAN AND MONETARIST VIEWS OF THE EFFECTIVENESS OF FISCAL AND MONETARY POLICY

Neo-Keynesians argue that Investment expenditure is relatively interest inelastic but that the speculative demand for money is interest elastic.

In contrast the monetarists argue that investment is interest elastic but that the speculative demand for money is relatively interest inelastic.

This implies that

NEO-KEYNESIANS		MONETARISTS
$MD_{2\bar{i}} = z_1$	$>$	$MD_{2\bar{i}} = z_2$

and

MONETARISTS NEO-KEYNESIANS

$I_{\overline{i}} = z_3 \quad > \quad I_{\overline{i}} = z_4$

From (5.3.3)

$$\frac{z_1}{z_1(S_{\overline{Y}} + T_{\overline{Y}}) + z_4 MD_{1\overline{Y}}} > \frac{z_2}{z_2(S_{\overline{Y}} + T_{\overline{Y}}) + z_3 MD_{1\overline{Y}}}$$

since $\frac{z_1}{z_4} > \frac{z_2}{z_3}$

so $\frac{\partial \overline{Y}}{\partial G_o}$ NEO-KEYNESIANS $>$ $\frac{\partial \overline{Y}}{\partial G_o}$ MONETARISTS

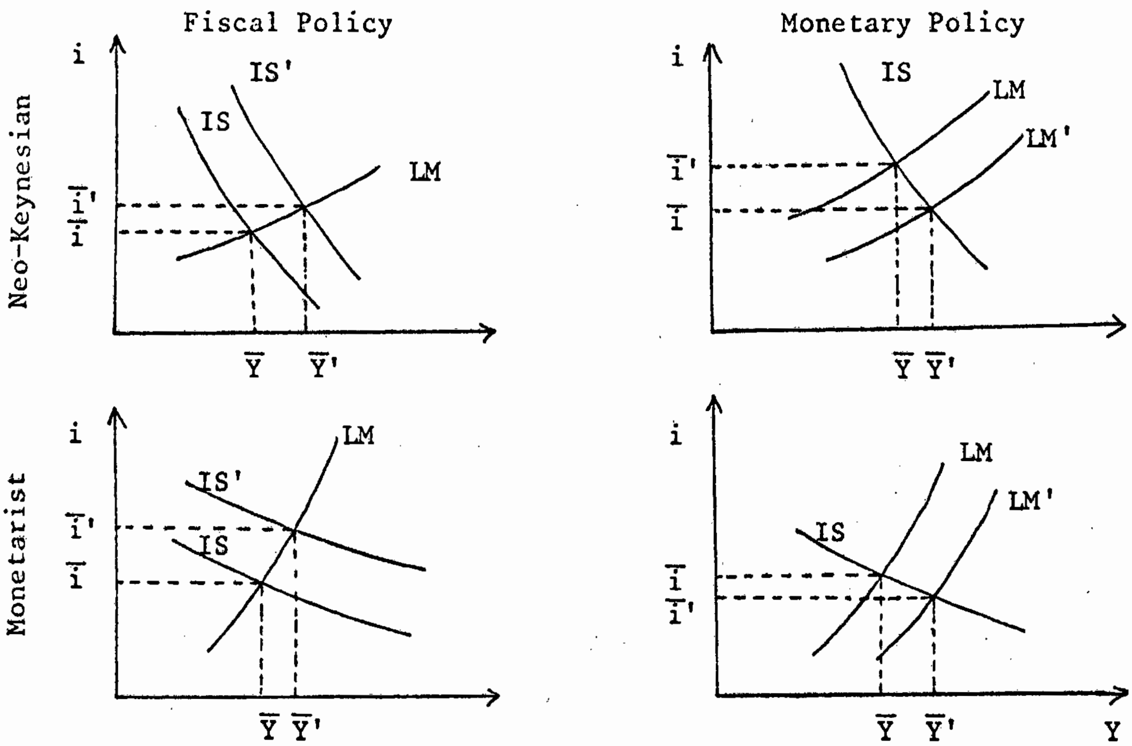
similarly it can be shown that

$\frac{\partial \overline{Y}}{\partial M_{s_o}}$ MONETARISTS $>$ $\frac{\partial \overline{Y}}{\partial M_{s_o}}$ NEO-KEYNESIANS

For the case of fiscal policy, therefore, Neo-Keynesians argue a greater effect on income than Monetarists. Conversely for monetary policy, Monetarists argue a greater effect on income than Neo-Keynesians.

The relative influence of fiscal and monetary policy is according to the analysis, dependent on the relative magnitudes of I_i and MD_{2_i}.

DIAGRAMATICALLY



We have, therefore, considered the major theoretical differences between the two schools of thought. These will in turn produce a divergence of policy emphasis regarding the implementation of stabilization policy.

For Monetarists, money is all important and primary importance should be allocated to monetary policy. Neo-Keynesians argue the usefulness of fiscal policy and de-emphasise the importance of money to some extent.

5.5 A NOTE ON THE TRANSMISSION MECHANISM

The transmission mechanism or monetary transmission mechanism is the manner in which changes in money bring about changes in income and prices through indirect effects in the goods market.

In the Neo-Keynesian framework an increase in money creates an imbalance in portfolios of financial assets because of excess cash. This cash is thus used to buy low risk liquid assets such as short term government stock, or corporate stock and equilibrium is regained via interest rate effects on Investment and thus income.

The crux of their theory is the fact that they focus attention on the interest rates of a range of assets that are highly liquid - this infers that in portfolio changes there is easily substitutability between money and these assets. This forces up the price of these assets and yields, and thus interest rates drop. The high price of stock brings forth new issues of stock and finance becomes available for investment purposes at low cost. Income is then expanded via the multiplier.

The monetarists argue that a rate of return can be imputed to all physical assets by equating the present value of the subjective flows of utility from these assets to their cost. Demand for these assets will be a function of the cost and opportunity cost of these purchases. Increasing the money supply will cause a downward shift in the yield of financial assets - opportunity costs of purchase of physical assets will drop and the level of aggregate demand will be affected directly.

5.6 MEASURES OF ECONOMIC ACTIVITY, FISCAL AND MONETARY ACTIONS

(i) ECONOMIC ACTIVITY

As a proxy for economic activity we use gross domestic product (G.D.P.) at current prices. G.D.P. comprises total spending of private, public and foreign sectors on services and locally produced goods.

(ii) MONETARY ACTIONS

The stock of money is taken to be a strategic variable in both the Neo-Keynesian and Monetarist schools. It is narrowly defined (M_1) as the private and public sector's holding of demand deposits with the Reserve Bank, Commercial Banks, National Finance Corporation, Discount Houses and Merchant Banks, plus cash (notes and coins). The wider definition of money (M_2) includes M_1 plus time deposits (short and medium term) with the above institutions.

The Monetarists have recently given particular attention to the variable money base (MB) where

$$MB = R + NDA$$

R = Reserves of the central bank (including gold)

NDA = Net Domestic Assets

Money base is assumed to be related to the money supply by a money multiplier (Brunner and Meltzer (1968)) which itself is assumed to be a function of interest rates, reserve requirements and economic activity.

The importance of the money base is not, however, given prominence in the Neo-Keynesian theory of income determination so in order to maintain generality money stock defined as M_2 was used as the indicator of monetary action.

(iii) FISCAL ACTIONS

In this study the influence of fiscal actions on economic activity is measured by Government Spending (G). G is assumed to be financed by tax receipts or public borrowing if no simultaneous expansion of M_2 occurs. Financing of G by monetary expansion will result in an accompanying increase of M_2.

5.7 *SPECIFICATION OF THE MODEL

To measure the relative strengths of fiscal and monetary actions the empirical relationship between Y, G and M_2 was established by the use of multiple regression. Since all time series have strong trends, and therefore, to avoid spurious correlations it was, from a theoretical aspect, considered more meaningful to examine the relationship between changes in Y and changes in G and M_2. The equation to be estimated was, therefore, of the form:

$$\Delta Y_t = f(\Delta G_t, \Delta G_{t-1}, \ldots, \Delta G_{t-n}; \Delta M_{2t}, \Delta M_{2t-1}, \ldots, \Delta M_{2t-n})$$

for some n, where $\Delta G_t = G_t - G_{t-1}$ (5.7.1)

$$\Delta M_{2t} = M_{2t} - M_{2t-1}$$

---

* Due to Andersen and Jordan (1968)

It is worth noting that the single equation (i.e. reduced form) model (5.7.1) used to test the relative strengths of fiscal and monetary policy has received some criticism - in particular from economists who argue the endogeneity of the money supply. Real Sector Models in structural form which incorporate the money supply as an endogenous variable have been developed by Sassanpour and Sheen (1976) and P.D. Jonson (1977). With reference to the Andersen and Jordan specification Ando and Modigliani in J.L. Stein (1976) comment that: "For a system as complex as the U.S. economy the St. Louis reduced form is a very unreliable method of estimating the true response path of nominal income to changes in monetary or fiscal aggregates," and with reference to the money supply: "..........the likelihood that at least at times during the period, the variable directly controlled by the monetary authorities was unborrowed or free reserves, or interest rates, ... (money) being then an endogenous variable." As a further problem they cite "The presence of correlation between the policy variables included in the reduced form regression and other policy and exogenous variables which effect nominal income, but are omitted in the reduced form."

In conclusion we might comment that this model cannot attempt to isolate the direct and indirect economic mechanisms by which fiscal and monetary action affect economic activity. However, it is able to measure the total response of economic activity to policy measures and this response will necessarily include all direct and indirect effects.

5.8 THE EMPIRICAL ANALYSIS

The method of Almon lags (see Appendix A) was used to estimate the lag structure of the relationship. The data used was seasonally adjusted quarterly data for the years 1960-1974.

We expect ΔY to be positively related to ΔM_2 and ΔG (at least initially), the magnitude of the coefficient indicating the strength of the relationship in an obvious way. ΔG is often, however, considered to affect Y negatively after some time interval because of the financial "crowding out" of the private sector. "Government expenditure financed from debt markets in competition with the private sector can possibly crowd out of the market an equal (or conceivably even greater) volume than would have financed private expenditure." (Andersen and Jordan (1968)).

5.8.1 ESTIMATION PROCEDURES

Two distinct assumptions were made about the distribution of the β's. We assume firstly that the β's can be approximated by a polynomial of degree 3 and no prior restrictions are placed on the distribution of the β's. In the second scheme we approximated the distribution of the β's by a polynomial of degree 3 but assume that the value of the β parameter of the last lagged variable, plus one equals zero. The assumption is that, a priori we expect that there is no influence on the change in Y through the independent variables after the specified time period, and hence any attempt to pick one up will produce spurious parameter estimates because of extraneous correlations.

Three different specifications were tried with an n of 3, 5 and 7. In each case a polynomial of degree 3 was used to estimate the coefficients.

5.8.2 INTERPRETATION OF THE PARAMETER ESTIMATES (TABLE 5.1)

It is seen immediately that in general the total response of ΔY to ΔM_2, i.e. SUM 1 is larger and more significant than the total response of ΔY to ΔG, i.e. SUM 2. The exception is when the lag distribution is taken up to seven quarters when SUM 1 becomes smaller than SUM 2. However, it seems that this is due to extraneous correlations picked up by the independent variables, because the sign of ΔM_2 becomes negative which is inconsistent with any theory. The R^2, although not satisfactory, are considered adequate when first differences rather than levels are used in the regressions. It is noted that the t-statistic for the coefficient relating to the response of ΔY at time t to ΔM_2 at time t is significant in each case and in some cases at time t-1. After that coefficients become smaller and lose significance. In the case of the coefficient relating the response ΔY to ΔG at time t, it is always positive but never significant. From time t-1 and onwards it becomes negative in almost all cases. This is consistent with the theory of "crowding out" where the initial response of ΔY to an increase in ΔG is positive, but then negative effects on private consumption expenditure cause this effect to tail off even to the point of causing a decrease in Y.

T A B L E 5.1

	n	ΔM	$ΔM_{-1}$	$ΔM_{-2}$	$ΔM_{-3}$	$ΔM_{-4}$	$ΔM_{-5}$	$ΔM_{-6}$	$ΔM_{-7}$	ΔG	$ΔG_{-1}$	$ΔG_{-2}$	$ΔG_{-3}$	$ΔG_{-4}$	$ΔG_{-5}$	$ΔG_{-6}$	$ΔG_{-7}$	CONST.	SUM_1	SUM_2	R^2	S.E.	D.O.F.
UNCONSTRAINED	3	0,3166	0,0363	0,0021	0,1993					0,3268	-0,5811	-1,6613	-1,2076					7,1328	1,1671	-3,1232	0,4551	94,7564	43,0
		2,4700	0,2884	0,0212	1,4105					6,3336	-0,5683	-1,5466	-1,1808					0,3126	3,9980	-1,4310			
	5	0,4852	0,0202	0,6591	0,2453	0,2221	-0,3672			1,1109	-0,7605	-0,2553	0,9321	1,0717	-1,5127			5,4486	0,6648	0,5773	0,5209	88,8461	43,0
		4,1228	0,1869	0,5904	2,1423	1,6031	-2,4235			1,1186	-0,8300	-0,3225	1,2064	1,2673	-1,4114			0,2405	2,1208	0,2329			
	7	0,3586	0,1918	0,1163	0,0897	0,0697	0,0142	-0,1192	-0,3727	0,8243	-0,2508	-0,2677	0,2937	0,9534	1,2315	0,6479	-1,2772	13,641	0,3484	2,1530	0,3257	105,4016	43,0
		2,5188	1,7784	0,9650	0,8193	0,5419	0,0891	-0,7719	-1,9547	0,7259	-0,2463	-0,2592	0,3434	1,1315	1,2613	0,6920	-0,9503	0,4561	0,7171	0,5590			
CONSTRAINED	3	0,4985	0,0506	-0,0022	-0,0707					1,2691	0,0693	-0,603	-0,6577					17,706	0,6176	0,0727	0,4211	107,5078	44,0
		3,7653	0,3547	-0,0193	0,4552					1,1792	0,0627	-0,707	-0,6014					0,6786	2,1403	0,0320			
	5	0,4319	0,1823	0,0532	0,0074	0,0080	0,0179			0,8198	0,0329	-0,2707	-0,2584	-0,0976	0,0443			9,0438	0,7006	0,2704	0,4572	100,1565	44,0
		3,0917	1,4387	0,4396	0,0652	0,0505	0,1165			0,6971	0,0304	-0,2753	-0,3310	-0,6058	0,0448			0,3160	1,7822	0,0876			
	7	2,7167	2,8122	1,0866	0,1202	-0,0745	-0,1383	-0,1383	-0,0985	0,9250	0,5720	0,3743	0,2881	0,2697	0,2754	0,2615	0,1843	6,4673	0,3579	3,1503	0,2915	119,3712	44,0
		2,7167	2,8122	1,0866	0,1202	-0,6904	-0,9388	-0,8848	-0,7977	0,7776	0,5785	0,3614	0,3257	0,3643	0,3275	0,2707	0,2375	0,2051	0,6970	0,7325			

The 5% (one-sided) significance point of the t-statistic with 45 degrees of freedom is 1,68.

M_2 is written here as M.

5.9 A SIMULATION STUDY

In order to clarify the implications of some of these results a simulation study was constructed whereby the effect of alternative government actions on income could be seen. Three alternative actions are considered.

(i) The rate of government spending is increased by one million rand and is financed by either borrowing from the public or increasing taxes.

(ii) The money stock is increased by one million rand with no adjustments to government expenditure.

(iii) The rate of government spending is increased by one million rand for one and a half years and is financed by increasing the money stock by an equal amount.

(This simulation analysis is due to Andersen and Jordan (1968) - apparently suggested by Milton Friedman!)

Consider the constrained equation with a lag structure up to t-5. This equation exhibits an adequate R^2 and the best t-statisticsin the set of regressions.

The impact on income of the first two actions may be measured by using the sums of the regression coefficients (SUM 1 and Sum 2). A million rand increase in the rate of government spending would, after six quarters, result in an increase of R270 thousand in Y. By comparison, an increase in money of one million rand would raise income by R700 thousand. The results of the last action are presented in Table 5.2 below. The rate (over 18 months) of government expenditure is increased by one million in the first quarter and held at that level for six quarters, e.g. if it were 20 million per quarter it now becomes 20,167 million per quarter. This requires an increase in money of 0,167 million in each quarter. Government expenditure is then reduced by one million so that we can consider the effects of financing the original increase in Government expenditure by increasing the money stock, otherwise money stock would have to continue to grow at the above rate. According to Table 5.2 Y rises by R700 thousand, this increase resulting entirely from monetary expansion.

5.10 CONCLUSIONS

This study has indicated that the Neo-Keynesian view in which fiscal measures are proposed as the controlling force for stabilization policy is ill-founded, and that over the period 1960-1974 monetary expansion had a larger impact on Y than fiscal measures of the same magnitude. This would support the monetarist view that much greater emphasis be placed on monetary measures in the conduct of stabilization policy.

These conclusions must, however, be seen in the light of the specification of the model which firstly, is strictly monetarist, assuming as it does the exogeneity of money and, secondly, is of a reduced form type (see the criticism by Ando and Modigliani above). It therefore does not, in the opinion of this author, necessarily detract from the use of Keynesian

TABLE 5.2

	INCREASE IN GOVERNMENT EXPENDITURE			REQUIRED INCREASE IN MONEY			TOTAL RESPONSE OF Y	
QUARTER	CHANGE IN EXP.	IMPACT EFFECT ON Y	CUMULATIVE EFFECT ON Y	CHANGE IN MONEY STOCK	IMPACT EFFECT ON Y	CUMULATIVE EFFECT ON Y	IMPACT EFFECT ON Y	CUMULATIVE EFFECT ON Y
1	1000	820	820	167	72	72	892	892
2	0	30	850	167	102	174	132	1024
3	0	-270	580	167	110	284	-160	864
4	0	-258	322	167	112	396	-146	718
5	0	-98	224	167	113	509	15	733
6	0	40	264	167	116	625	156	889
7	-1000	-820	-556	0	44	669	-776	113
8	0	-30	-586	0	14	683	-16	97
9	0	270	-316	0	6	689	276	373
10	0	258	-58	0	4	693	262	635
11	0	98	40	0	3	696	101	736
12	0	-40	0	0	0	696	-40	696

models in real sector analysis. Although it is essential to recognise the importance of money in real sector analysis, it is felt that this is best done by endogenizing the money supply (or money base). The "spillover" effects of money on domestic expenditure may then be determined by the excess demand for real balances (see for example Sheen (1976)). We may therefore reaffirm the conclusions of Chapter 4 which stress that the development of a realistic model of the whole economy must involve endogenizing the monetary sector.

APPENDIX A

STATISTICAL AND MATHEMATICAL THEORY

INTRODUCTION

In this Appendix an account is given of the statistical and mathematical theory that is pertinent to this work. We first present a review of the Multiple Linear Regression Model, its assumptions and associated summary statistics. This leads on to a discussion of the problems associated with violations of these assumptions — in particular a detailed development of the problem of multicollinearity is given , a topic that has received very little attention in basic econometric textbooks to date. A discussion of parameter estimation in simultaneous systems by the method of Full Information Maximum Likelihood (with reference to the Newton Raphson iterative procedure) is then given. Finally we include a note on the stability criteria of certain differential equation systems that are particularly relevant to this work.

A.1 THE LINEAR REGRESSION MODEL

We assume Y is linearly related to p explanatory variables $X_2, X_3, \ldots, X_p$ and an error term u. If we have a sample of n observations on Y and $X_2, X_3, \ldots, X_p$ we may write (t subscript refers to time)

$$Y_t = \beta_1 + \beta_2 X_{2t} + \ldots \beta_k X_{pt} + u_t \quad (t = 1, \ldots, n).$$

The β_i ; $i = 1, \ldots, p$ are constant regression coefficients.

This system may be rewritten as

$$Y = X\beta + u$$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} \qquad X = \begin{bmatrix} 1 & X_{21} & \cdots & X_{p1} \\ 1 & X_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & X_{2n} & & X_{pn} \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

<u>The following assumptions are made:</u>

(1) u is distributed as a real, normal, random variable with

$E(u) = 0 \quad E(uu') = I_n$, I_n an $(n \times n)$ identity matrix

(2) u is independent of the explanatory variables, or

$E(X'u) = 0$

This is equivalent to the assumption that X is non stochastic.

(3) X has rank $p \leqslant n$.

i.e. The columns of X must not be collinear.

(4) We assume that the X are free of measurement errors.

(5) The relationship is correctly specified, i.e. all the important regressors have been included explicitly in the model, which has a correct mathematical form.

If we define $\underset{(n\times 1)}{e} = Y - \hat{Y}$ where $\hat{Y}$ is some estimate of Y i.e. $\hat{Y} = \hat{\beta}X$, then minimization of $e'e$

yields

$$\hat{\beta} = (X'X)^{-1}X'Y \ .$$

This is the least squares estimate of β . It can be shown (Johnston (1963)) that

(i) $E(\hat{\beta}) = \beta$

(ii) $Var(\hat{\beta}) = \sigma^2(X'X)^{-1}$

(iii) $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$

(iv) $S^2 = \dfrac{e'e}{n - p}$

(v) $E(S^2) = \sigma^2$.

We have that $\hat{\beta}_i \sim N\ (\beta_i, \operatorname{diag}_i \sigma^2(X'X)^{-1})$, $i = 2,\ldots,p$.

To test the hypothesis $H_0 : \beta_i = 0$

$H_1 : \beta_i \neq 0$

we use the t-statistic $= \dfrac{\hat{\beta}_i}{\operatorname{diag}_i S^2 \ (X'X)^{-1}}$.

<u>This has a t-distribution with (n - p) degrees of freedom.</u>

The proportion of variation in Y explained by the X is given by <u>the multiple coefficient of determination — R^2</u> .

$$R^2 = \frac{\text{explained sum of squares}}{\text{total sum of squares}} = \frac{\sum_{t=1}^{n} (\hat{Y}_t - \overline{Y})^2}{\sum_{t=1}^{n} (Y_t - \overline{Y})^2}$$

$$= \frac{\hat{\beta}'X'Y - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)}{Y'Y - \frac{1}{n}\left(\sum_{t=1}^{n} Y_t\right)} \ .$$

R^2 is usually adjusted for degrees of freedom since addition of explanatory variables always increases R^2 but their contribution is not necessarily significant.

$$R^2_{\substack{\text{adjusted for } p \\ \text{degrees of freedom}}} = \overline{R^2} = \frac{1-p}{n-p} + \frac{n-1}{n-p} \cdot R^2 .$$

VIOLATION OF THE ASSUMPTIONS

A.2 THE PROBLEM OF AUTOCORRELATION

If $E(uu') = \sigma^2\Omega$

and $\Omega \neq I$ we have violated assumption (1) and autocorrelation is present in the model.

A.2.1 SOURCES OF AUTOCORRELATION

(i) Mis-specification of the mathematical model — If for example we consider a linear relationship between Y and X when the true relationship is log-linear.

(ii) Deviation of y from its expected value because of uncontrollable factors such as political uncertainty, during some period. These should be adjusted for by dummy variables.

(iii) Omitted explanatory variables.

A.2.2 EFFECTS OF AUTOCORRELATION

(i) $E(\hat{\beta}) = \beta$ if $\hat{\beta} = (X'X)^{-1} X'Y$ is used (estimator still unbiased)

(ii) $Var(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X(X'X)^{-1}$

$> \sigma^2 (X'X)^{-1}$

$\sigma^2 (X'X)^{-1}$ thus underestimates $Var(\hat{\beta})$.

If Ω is known it can be shown that the unbiased linear estimator of β with least variance is

$$\hat{\beta}^* = (X'\Omega^{-1}X)^{-1} X'\Omega Y \ .$$

This is known as the Generalized Least Square estimator.

A.1.2.3 TEST FOR AUTOCORRELATION

We test that the error process follows some auto-correlation scheme. The simplest scheme is that of the first-order autocorrelation process.

We assume

$$u_t = \rho u_{t-1} + \varepsilon_t \quad \text{with} \quad |\rho| < 1$$

with $E(\varepsilon_t) = 0 \quad E(\varepsilon\varepsilon') = 0 \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$.

A test for first-order autocorrelation has been given by Durbin and Watson (1950, 1951).

The null-hypothesis $H_0 : \rho = 0$

against $H_1 : \rho > 0$

or $H_1 : \rho < 0$

is tested using the statistic

$$d^* = \frac{\sum_{t=2}^{n} (e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$$

the Durbin-Watson statistic.

d^* is then compared with the d_L and d_U values of the Durbin-Watson tables (with $n - p$ degrees of freedom).

(1) For the test $H_0 : \rho = 0$

against $H_1 : \rho > 0$

the following applies

(i) If $d^* \geqslant du$ we accept H_0 .

(ii) If $d^* \leqslant d_L$ we accept H_1 .

(iii) If $d_L < d^* < d_U$ the test is inconclusive.

(2) For the test $H_0 : \rho = 0$

$H_1 : \rho < 0$

(i) If $4 - d^* \geqslant d$ we accept H_0 .

(ii) If $4 - d^* \leqslant d$ we accept H_1 .

(iii) If $d_L < 4 - d^* < d_U$ the test is inconclusive.

A.2.4 THE CASE OF A LAGGED ENDOGENOUS VARIABLE

In Chapter 2 we were led to consider an equation of the form

$$Imp_t = \beta_0 + \beta_1 Imp_{t-1} + \beta_2 Gde_t + u_t \ .$$

The assumption of full independence between the explanatory variables and the error term is untenable. Clearly Imp_{t-1} is dependent on u_{t-1} and hence on u_{t-2} etc. - in fact on all prior disturbances $u_{t-1}, u_{t-2}, \ldots, u_{t-n}$. As a consequence the O.L.S. estimation $\hat{\beta}$ is biased:

$$E(\hat{\beta}) = \beta - \frac{2\beta}{n} + 0(n^{-2}) \ ,$$

but consistent. (Johnston (1963))

If the u_t are related by a first order scheme as discussed above it can be shown (Johnston (1963)) that

$$\text{plim } \hat{\beta} - \beta = \frac{(1 - \beta^2)}{1 + \beta\rho} \quad .$$

It can also be shown (Johnston (1963)) that the Durbin-Watson statistic is biased towards 2 in the above situation.

A large-sample test for serial correlation when lagged dependent variables are present as independent variables has been developed by Durbin (1970) .

Defining $\hat{\rho} = \dfrac{\sum_{t=2}^{n} e_t \, e_{t-1}}{\sum_{t=1}^{n} e_t^2}$

we compute $h = \hat{\rho} \sqrt{\dfrac{n}{1 - n \hat{\text{Var}} (\hat{\beta_1})}}$

where $\hat{\text{Var}}(\hat{\beta}_1)$ is the estimate of the sample variance of $\hat{\beta}_1$, the coefficient of the lagged dependent variable in the regression.

Durbin showed that for $(n > 50)$

$$h \sim N(0,1) \; .$$

We would therefore reject $H_0 : \rho = 0$ at the 5% level.

if $|h| > 1.64$.

A.2.5 ADJUSTMENT FOR AUTOCORRELATION

If first order autocorrelation is detected, the following iterative procedure may be used to obtain least square estimates.

For purpose of explanation consider the model

$$Y_t = \beta_0 + \beta_1 X_t + u_t \ ; \quad t = 1,\ldots,n$$

with $u_t = \rho u_{t-1} = e_t \qquad |\rho| < 1$

and $E(e) = 0 \quad E(ee') = \sigma^2 I_n$.

We have

$$Y_t - \rho Y_{t-1} = \beta_0(1 - \rho) + \beta_1(X_t - \rho X_{t-1}) + e_t \ .$$

Denoting estimates of β_0, β_1 and ρ by $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\rho}$, the sum of squares of the error is given by

$$\sum_{t=1}^{n} e_t^2 = \sum_{t=1}^{n} [(Y_t - \hat{\rho}Y_{t-1}) - \hat{\beta}_0(1 - \hat{\rho}) - \hat{\beta}_1(X_t - \hat{\rho}X_{t-1})].$$

The direct minimization of the above is impossible because it leads to non-linear equations and so linear expressions for $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\rho}$ cannot be obtained (Johnston (1963)).

<u>The Cochrane-Orcutt iterative procedure</u> (Cochrane-Orcutt (1949)) is used to circumvent this problem. Starting with an arbitrary value of $\hat{\rho}$ say $\hat{\rho}'$ the sum of squares is minimized with respect to the parameters $\hat{\beta}_0$ and $\hat{\beta}_1$ to obtain $\hat{\beta}_0'$ and $\hat{\beta}_1'$. Then keeping $\hat{\beta}_0$ and $\hat{\beta}_1$ fixed at $\hat{\beta}_0'$ and $\hat{\beta}_1'$ the sums of squares are minimized with respect to $\hat{\rho}$ obtaining a new value $\hat{\rho}''$. Keeping this fixed in turn minimize once again, with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$ obtaining $\hat{\beta}_0''$ and $\hat{\beta}_1''$. This iterative procedure is continued until successive values differ by some fixed amount.

The AUTO package on the UNIVAC 1106 at U.C.T.

provides for the use of the Cochrane-Orcutt iterative method and was used to adjust for autocorrelation when it was found to be present.

A.3 THE PROBLEM OF MULTICOLLINEARITY

One of the basic assumptions of the linear model (see above) is that the data matrix X $(n \times p)$ has rank p i.e. that no linear dependence exists between the explanatory variables (otherwise $X'X$ would be singular). A less extreme but still very serious case arises when the assumption is only just satisfied, that is when some or all of the explanatory variables are highly but not perfectly collinear, i.e. the prediction vectors are non-orthogonal but not dependent. This is of course an extremely common phenomenon with economic variables, for example in the prediction of imports, imports lagged one and gross domestic expenditure are highly correlated. Two methods have recently been proposed for the solution of such problems of multicollinearity — these are discussed below.

A.3.1 THE RIDGE SOLUTION

Hoerl and Kennard (1970) have proposed a method to deal with non-orthogonal problems which can actually decrease the mean square error of estimation.

Consider the standard model

$Y = X\beta + e$ where X is $(n \times p)$; with rank $p < n$ and β is $(p \times 1)$.

$$E(e) = 0$$

$$E(ee') = \sigma^2 I_n \; . \qquad \text{(A.3.1.1)}$$

The ordinary least square estimator is (Johnston (1963))

$\hat{\beta} = (X'X)^{-1}X'Y$ where X and Y may be expressed in standardized form i.e. by subtracting sample means and dividing by sample standard deviations. $X'X$ will then be the correlation matrix.

This estimator minimizes the sums of squares of the residuals

$$\phi(\hat{\beta}) = (Y - X\hat{\beta})'(Y - X\hat{\beta}) \; .$$

We consider here the case where $X'X$ is somewhat different from a unit matrix (violation of assumption (3)) so that the X vectors are non-orthogonal.

To demonstrate the effects of this condition let

L_1 = distance from $\hat{\beta}$ to β

$$\therefore (L_1)^2 = (\hat{\beta} - \beta)'(\hat{\beta} - \beta) \; .$$

Denote the ranked eigenvalues (largest to smallest) of $X'X$ by λ_i ; $i = 1,\ldots,p$.

It can then be shown (Hoerl and Kennard (1970)) that

$$E(L_1{}^2) = \sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}$$

$$\mathrm{Var}(L_1{}^2) = 2\sigma^4 \sum_{i=1}^{p} \left(\frac{1}{\lambda_i}\right)^2 \; .$$

Lower bounds for the above are σ^2/λ_p and $2\sigma^4/\lambda_p$ respectively so an $X'X$ matrix possessing one or more small eigenvalues will tend to yield large L_1 .

Hoerl and Kennard (1970) suggested that to control the inflation of variance and general instability associated with the least square stimates one might use

$$\hat{\beta}^* = (X'X + kI)^{-1}X'Y \; ; \quad k \text{ a constant scalar} \geqslant 0 \, .$$

They note that the family of estimate given for $k \geqslant 0$ bear many mathematical similarities to the portrayal of quadratic response functions. For this reason they label estimation procedures built around $\hat{\beta}^*$ as "ridge regression". $\hat{\beta}^*$ is the ridge estimator.

We note that $\hat{\beta}^* = \left(X'X(I_p + k(X'X)^{-1})\right)^{-1} X'Y$

$$= (I_p + k(X'X)^{-1})^{-1}(X'X)^{-1}X'Y$$

$$= Z\hat{\beta}$$

where $\quad Z = (I_p + k(X'X)^{-1})^{-1}$.

Consider now the mean square error of $\hat{\beta}^*$ defined as

$$E[(\hat{\beta}^* - \beta)'(\hat{\beta}^* - \beta)] \, .$$

Note: The mean square error of an estimator $\hat{\alpha}$ of α

$$= E[(\hat{\alpha} - \alpha)'(\hat{\alpha} - \alpha)]$$

$$= E\Big[(\hat{\alpha} - E(\hat{\alpha}))'(\hat{\alpha} - E(\hat{\alpha})) + E\,(E(\hat{\alpha}) - \alpha)'(E(\hat{\alpha}) - \alpha)\Big]$$

$$+ 0$$

$$= \text{Var}\,\hat{\alpha} + (\text{Bias}\,\hat{\alpha})^2 \, .$$

$$E[(\hat{\beta}^* - \beta)'(\hat{\beta}^* - \beta)]$$

$$= E[(\hat{\beta} - \beta)'Z'Z(\hat{\beta} - \beta) + (Z\beta - \beta)'(Z\beta - \beta)]$$

$$= \sigma^2 \,\text{Trace}\,(X'X)^{-1}Z'Z + \beta'(Z - I)'(Z - I)\beta$$

$$= \sigma^2\,[\text{Trace}\,(X'X + kI)^{-1} - k\,\text{Trace}\,(X'X + kI)^{-2}]$$

$$+ k^2\beta'(X'X + kI)\beta$$

$$= \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} + k^2\beta'(X'X + kI)$$

$$= \gamma_1(k) + \gamma_2(k) \ .$$

The term $\gamma_1(k)$ can be shown (Hoerl and Kennard (1970)) to be the sum of the variances (total variance) of the ridge estimates. The second element $\gamma_2(k)$ is the square distance from $Z\beta$ to β (see above). It will be zero when $k = 0$ since Z is then equal to I . Thus $\gamma_2(k)$ can be considered to be the square of a bias introduced when $\hat{\beta}^*$ is used rather than $\hat{\beta}$.

The figure below shows the relationship between the variances, the squared bias and the parameter k .

Mean Square Error Function

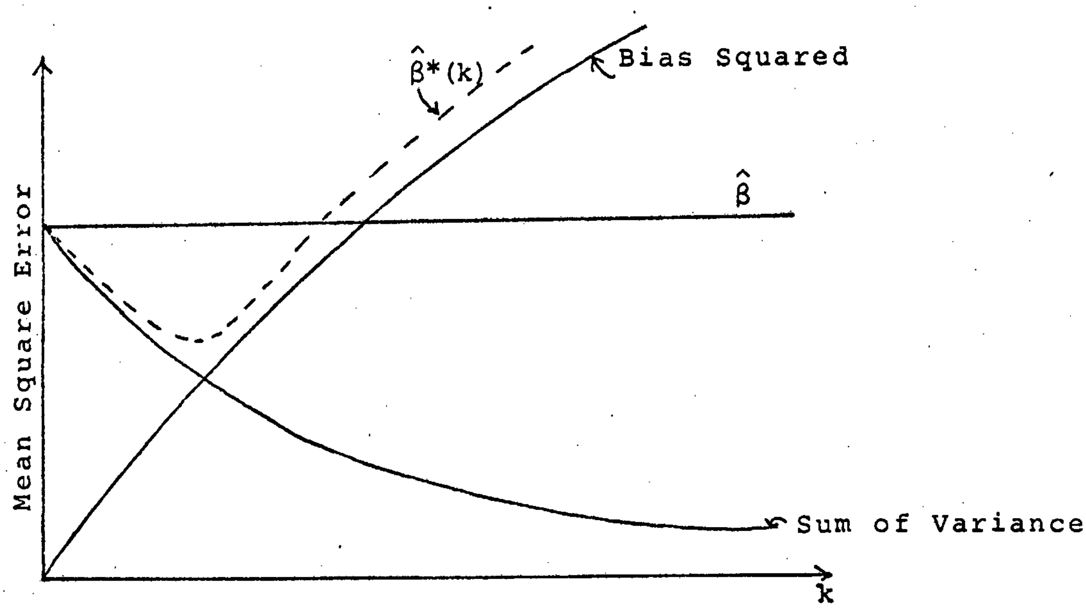


Figure A1

It is seen that the total variance decreases as k increases while the squared bias increases with increasing k . As is indicated by the dotted line which is

$$E[L_1{}^2(k)] = \gamma_1(k) + \gamma_2(k) \ ,$$

the possibility exists that there are values of k for which the mean square error is less for $\hat{\beta}^*$ than it is for $\hat{\beta}$. This possibility is supported by the mathematical properties of $\gamma_1(k)$ and $\gamma_2(k)$. The function $\gamma_1(k)$ is a monotonic decreasing function of k, while $\gamma_2(k)$ is monotonic increasing (Hoerl and Kennard (1970)). However, the most significant feature is the value of the derivative of each function in the neighbourhood of the origin. These derivatives are:

$$\lim_{k \to 0^+} \frac{d\gamma_1}{dk} = -2\sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}$$

and $$\lim_{k \to 0^+} \frac{d\gamma_2}{dk} = 0 \ .$$

(Hoerl and Kennard (1970))

Thus, $\gamma_1(k)$ has a negative derivative which approaches $-2p\sigma^2$ as $k \to 0^+$ for an orthogonal $X'X$ and approaches $-\infty$ as $X'X$ becomes ill-conditioned and $\lambda_p \to 0$.

However as $k \to 0^+$ we see that $\gamma_2(k)$ is flat at the origin. These properties lead to the conclusion that it is possible to move to $k > 0$, take a little bias and substantially reduce the variance, thereby improving the mean square error of estimation and prediction.

Hoerl and Kennard (1970) state that based on their experience the following can be used to select a single value of k and thus give rise to a "best" $\hat{\beta}^*$.

(1) At a certain value of k the system will stabilize and have the general characteristics of an orthogonal system.

(2) Coefficients with apparently incorrect signs at $k = 0$ will have changed to have the proper sign.

(3) The residual sum of squares will not have been inflated to an unreasonable value i.e. R^2 will still be reasonable.

It should be noted, however, that even in the case of an orthogonal system increasing k stabilises the system and drives all coefficients to zero since

$$\hat{\beta}^* = (X'X + kI)^{-1}X'Y \xrightarrow[k]{} \frac{1}{(k+1)} X'Y \to 0 \qquad (X'X = I)$$

In the case of $X'X \neq I$ we obtain

$$\hat{\beta} = (X'X + kI)^{-1}X'Y \xrightarrow[k]{} \frac{1}{k} X'Y \to 0 \ .$$

Use of large k will thus necessarily tend to swamp any meaningful results. It is therefore often difficult to disentangle the opposing effects of swamping toward zero and stabilisation. Further comment is made in relation to these ideas in Chapter 2, when the above methods are applied to the equation in hand.

A problem of great importance, which has not been examined in the literature, is the reliability of the ridge estimates. Hoerl and Kennard's method of taking k at which the ridge traces become stable is reasonable but subject to the criticism above. It was thought* that it might be more reasonable to simply compare total variance $\gamma_1(k)$ with R^2, and trade off decreasing variance with decreasing R^2. It was shown, however*, that the two

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* Refer to "A Consumption Expenditure Model for Ten Selected Commodities in the South African Economy", G.D.I. Barr (1976) B.Sc. Hons. project.

methods produce basically equivalent results when cognizance is taken of the fact that almost all coefficient stabilization takes place in the range $k = 0.0$ to 0.3 .

A.3.2 THE CHARACTERISTIC ROOT SOLUTION

Consider again the model (A.3.1.1) .

Let $Z = (Y;X)$ and $S = \begin{pmatrix} Y'Y & Y'X \\ X'Y & X'X \end{pmatrix} = Z'Z$

i.e. S is the matrix of correlations between dependent and independent variables.

Let λj ; $j = 0,1,\ldots,p$ be the eigenvalues of S .

Let γj ; $j = 0,1,\ldots,p$ be the corresponding eigenvectors of S .

$$\gamma j = \begin{pmatrix} \gamma_0 j \\ \gamma_1 j \\ \vdots \\ \gamma_p j \end{pmatrix} \qquad \text{Define } \gamma^0 j = \begin{pmatrix} \gamma_1 j \\ \gamma_2 j \\ \vdots \\ \gamma_p j \end{pmatrix} .$$

$$\text{Let } \Gamma = (\gamma_0, \gamma_1, \ldots, \gamma_p) .$$

$$\Lambda = \text{diag } (\lambda_0, \lambda_1, \ldots, \lambda_p)$$

$$\Gamma' S \Gamma = \Lambda$$

$$\text{or } \lambda_j = \gamma_j' S \gamma_j \; ; \quad j = 0,1,\ldots p$$

$$= \sum_{i=1}^{n} (Y_i \gamma_0 j + \sum_{r=1}^{p} X_{ir} \gamma_{rj})^2 .$$

If any $\lambda_j = 0$ then,

$$(Y_i \gamma_0 j + \sum_{r=1}^{p} X_{ir} \gamma_{rj}) = 0 .$$

Denoting $\hat{Y}_{ij}$ as the i^{th} component of the predictor $\hat{Y}_j$ obtained from $\lambda_j = 0$ we may obtain

$$\hat{Y}_{ij} = -\gamma_{0j}^{-1} \sum_{r=1}^{p} X_{ir} \gamma_{rj} \quad (\gamma_{0j} \neq 0) \quad (A.3.2.1)$$

That is, an exact linear dependence exists between the columns of Z and we have a perfect predictor for the vector Y .

If $\lambda_j = 0$ and $\gamma_{0j} = 0$

then $\sum_{r=1}^{p} X_{ir} \gamma_{rj} = 0$

and an exact linear dependence exists amongst the columns of X which implies multicollinearity. For a predictor of the form (A.3.2.1) the residual sum of squares

$$= \sum_{i=1}^{n} (Y_i - \hat{Y}_{ij})^2$$

$$= \frac{\lambda_j}{\gamma_{0j}}$$ (Gunst, Webster and Mason (1974)).

Normally none of the predictor vectors $\hat{Y}_j$ will by itself be a good predictor. We therefore consider linear combinations of these predictors in particular consider

$$\hat{Y} = \sum_{j=0}^{p} a_j \gamma_{0j} \hat{Y}_j \quad \text{where} \quad \sum_{j=0}^{p} a_j \gamma_{0j} = 1$$

Now $\quad \hat{Y}_j = -\gamma_{0j}^{-1} X\gamma_j^0$

So $\quad \hat{Y} = -X \sum_{j=0}^{p} a_j\gamma_j^0 = X\bar{\beta}$

$\quad \bar{\beta} = -\sum_{j=0}^{p} a_j\gamma_j^0$ (the characteristic root predictor vector).

The residual sum of squares $= (Y - \hat{Y})'(Y - \hat{Y})$

$$= \sum_{j=0}^{p} a_j^2\lambda_j .$$

The Least Squares and Modified Least Squares Estimators

If in the residual sum of squares above the a_j are chosen to minimize the sums of squares we must arrive at the ordinary least squares predictor.

We wish to minimise $\sum_{j=0}^{p} a_j{}^2\lambda_j$.

subject to $\sum_{j=0}^{p} a_j\gamma_{0j} = 1$.

The Lagrange function L is thus ($2\mu_0$ being the Lagrange multiplier)

$$L = \sum_{j=0}^{p} a_j{}^2\lambda_j - 2\mu_0 \left(\sum_{j=0}^{p} \gamma_{0}j\ a_j - 1\right)$$

$$\frac{\partial L}{\partial a_j} = 2a_j\lambda_j - 2\mu_0\gamma_{0}j = 0$$

$$\therefore \quad a_j = \frac{\mu_0\gamma_0 j}{\lambda_j} \ .$$

We know $\sum_{j=0}^{p} \gamma_{0j}a_j = 1$ so $\sum_{j=0}^{p} \frac{\gamma_{0j}\ \mu_0\ \gamma_{0j}}{\lambda_j} = 1$

$$\therefore \quad \mu_0 = \frac{1}{\sum_{j=0}^{p} \gamma_{0j}{}^2/\lambda_j} \ .$$

The residual sum of squares $= \sum_{j=0}^{p} a_j{}^2\lambda_j$

$$= \sum_{j=0}^{p} \frac{\gamma_0 j^2\lambda}{\lambda} \left(\sum_{j=0}^{p} \frac{\gamma_0 j^2}{\lambda j}\right)^{-2}$$

$$= \left(\sum_{j=0}^{p} \gamma_{0j}{}^2/\lambda_j\right)^{-1}$$

$$= \mu_0 \ .$$

The Least Squares Estimator

$$\hat{\beta} = \hat{\bar{\beta}} = -\sum_{j=0}^{p} a_j \gamma_j{}^0$$

$$= -\sum_{j=0}^{p} \frac{\gamma_{0j}}{\lambda_j} \gamma_j{}^0 \left(\sum_{j=0}^{p} \gamma_{0j}{}^2/\lambda_j \right)^{-1} \qquad (A.3.2.1)\ .$$

Of crucial importance when discussing near singularities of S is whether these singularities contain information about the underlying model $Y = X\beta + e$ i.e. whether these singularities are predictive or non-predictive. Or, in other words, whether the singularity is caused by high correlation between Y and some X variable or between the X variables themselves.

Geometrically, one obtains a clearer picture. Consider the n data points $(Y_i, X_{1i}, \ldots, X_{ip})$; $i = 1,\ldots,n$ as n points in $p+1$ dimensional Euclidean space defined by mutually orthogonal axes $(Y, X_1, \ldots, X_p)$. The characteristic vectors of S define a set of mutually orthogonal axes $(Z_0, Z_1, \ldots, Z_p)$. The γ_j are normalized and the elements $(\gamma_{0j}, \gamma_{1j}, \ldots, \gamma_{pj})$ represent the cosine of the angles between Z and $(Y, X_1, \ldots, X_p)$.

The characteristic root corresponding to a particular γ_j measures the spread of the n data points in the direction defined by γ_j since

$$\lambda_j = \sum_{i=1}^{n} \left(Y_i \gamma_{0j} + \sum_{r=1}^{p} X_{ir} \gamma_{rj} \right)^2$$

i.e. the squared projections of the n data points on γ_j .
A small value of λ_j indicates that there is little variation in the Z_j direction and that singularity exists in S .

We clearly want to preserve high correlation bonds between Y and X variables as these lead to good prediction (a predictive singularity). If however γ_{0j} is near zero ($\cos\theta \doteq 0 \Rightarrow \theta \doteq 90^\circ$; θ the angle between Z_j and Y), Z_j is nearly orthogonal to the Y axis and the singularity is non-predictive. If both γ_{0j} and λ_j are small the γ_j reveals a non-predictive near singularity i.e. the singularity is in the $X'X$ matrix. The least squares estimator (A.3.2.1) is a linear combination of all $p+1$ characteristic vectors including characteristic vectors corresponding to non-predictive near-singularities. The modified least squares estimator which follows (A.3.2.2) utilises only linear combinations not having both λ_j and γ_{0j} small. In this way the estimates are adjusted for the effect of non-predictive near singularities.

Suppose $\gamma_0, \ldots, \gamma_{q-1}$ correspond to non-predictive near singularities i.e. $\lambda_0, \ldots, \lambda_{q-1}$ and $\gamma_{00}, \ldots, \gamma_{0q-1}$ are close to zero. The least squares estimator (A.3.2.1) can be adjusted by setting $a_0 = a_1 = \ldots = a_{q-1} = 0$.
This yields an error sum of squares

$$= \frac{1}{\sum_{j=q}^{p} \gamma_{0j}^2/\lambda_j} \quad .$$

and the modified least squares estimator

$$\hat{\beta}_{(0,\ldots,q-1)} = -\sum_{j=q}^{p} \gamma_{0j}\, \gamma_j{}^0/\lambda_j \left(\sum_{j=q}^{p} \gamma_{0j}{}^2/\lambda_j\right)^{-1} \qquad \text{(A.3.2.2)}.$$

The estimators (A.3.2.1) and (A.3.2.2) are often very different; firstly because λ_j close to zero result in unreliable estimates $\hat{\beta}$ with large variance, and secondly the a_j corresponding to characteristic vectors revealing non-predictive near singularities are often large relative to the remaining a_j. When this occurs the terms $a_j\,\gamma^0{}_j$ $(j = 0,\ldots,q-1)$ can dominate $\hat{\beta}$. Removing these dominating terms will yield more accurate estimates of the true vector β.

A.3.3 CONCLUSIONS AND COMPARISONS OF THE APPLICABILITY OF RIDGE AND CHARACTERISTIC ROOT REGRESSION

Unfortunately with modified least square estimation (M.L.S.) the only indication of the reliability of the estimate is, as with the ridge procedure, observation of the decrease in variance and its trade-off with the decrease in R^2.

Gunst, Webster and Mason (1976) have noted that theoretical comparison of M.L.S. with other methods is very difficult since the distributional properties of the M.L.S. estimates are unknown due to the complicated multivariate random variables involved.

If $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_p$ are the eigenvalues of the

S matrix they state that by analogy, with principal components regression on the latent vectors of $X'X$, that if γ_0 is the eigenvector corresponding to a near predictive singularity i.e. λ_0 and γ_{00} are close to 0, the sums of variance of the M.L.S. estimate is given approximately by: $\sigma^2 \sum_{i=1}^{p} \frac{1}{\lambda_i}$.

They also derive the formula $\zeta^2 \left(\sum_{j=1}^{p} \gamma_{0j}^2/\lambda_j\right)^{-1}$

where ζ^2 is a constant for the residual sum of squares for the modified least squares estimates, where λ_0 and γ_{00} correspond to the eigen-value and eigenvector revealing a non-predictive near singularity. (Note that the residual sum of squares for O.L.S. is given by

$$\zeta^2 \left(\sum_{j=0}^{p} \gamma_{0j}^2/\lambda_j\right)^{-1} .$$

In order to make a comparison between ridge and M.L.S. estimation a simulation study* was made of data (constructed so that it was highly collinear with respect to the X's) published by Gunst, Webster and Mason (1974) where the augmented correlation matrix (S) had the following ranked eigen-value, y component eigenvector spectrum for a model of the form

$$y = \sum_{i=1}^{6} \beta_i x_i .$$

* Using a program written by the author - see Appendix C.

j	0	1	2	3	4	5	6
λ_j	0,001	0,0287	0,3115	0,9178	1,1150	2,1816	2,444
γ_{0j}	0,0339	0,6987,	0,0388	0,0388	0,3406	0,6006	0,1653

The following table sumarizes the results.

	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$	Sums of Variance
True Values	2,0000	1,0000	0,3000	-2,0000	3,0000	10,0000	
O.L.S. estimates	-6,0378	-8,4720	-10,1435	-11,7271	4,0967	9,4056	$\sigma^2 1037,6$
M.L.S. estimates (excluding λ_0/γ_0)	2,5447	-0,3982	0,2416	-0,7348	4,2125	9,4914	$\sigma^2 38,23$
Ridge estimates							
k=0,1	1,19124	-0,8043	-0,0054	-1.3218	3,9026	9,4318	$\sigma^2 29,47$
k=0,2	1,79653	-0 -0,7431	0,2041	-1,2925	3,6026	7,6117	$\sigma^2 19,62$
k=0,3	1,68041	-0,6991	0,3217	-1,2818	3,3309	6,9501	$\sigma^2 12,92$

With ridge regression it is seen that almost all significant change is taken up with an initial k of 0,1 and the stabilisation of the coefficient is often in a direction away from that of the true coefficient. Therefore, although it is obviously very beneficial in cases of multi-collinearity, larger k than say 0,3 should only be used with due realization of the possible swamping effect.

Comparison between M.L.S. and ridge regression coefficients suggest that both are reasonable methods, with little to choose between M.L.S. and ridge with sensible k . It seems that the shape of the data space might have a significant effect on their relative merits but this has not been investigated. With the regressions performed below it turned out that M.L.S. was never applicable using the criteria of Gunst and Mason - rejecting λ_j and γ_{0j} in the estimator when they are less than 0,05 and 0,1 respectively. In the equation describing labour (Chapter 2) the use of the method and its effect using various rejection criteria is demonstrated.

It should be noted that in the simulation under consideration the data portrayed marked singularity in the $X'X$ matrix with a determinant of the order of 10^{-6} . The effects of introducing k were thus remarkable and it was clear that there was dramatic improvement in the coefficients. Stabilization was easy to detect, in fact picking a reasonable k that was guaranteed to improve our estimates was not difficult. In general, however, the cases that are dealt with in Chapter 2 and warranted adjustment exhibited much "less singular" matrices and therefore to disentangle the positive effects of increasing k from its swamping effect was more difficult.

It seems though that whenever multicollinearity was present some small k will always yield an estimate with smaller mean square error than O.L.S. (we have an analytical proof of this since the mean square error decreases initially); the optimum value of k is extremely dependent however on the singularity of our data.

A.4 THE LAG STRUCTURE SCHEME OF ALMON

As discussed above, economic theory often suggests that the influence of an explanatory variable is spread over several time periods. For example, consumption at time t was considered to be dependent on disposable income at time $t, t-1, t-2$ etc. We therefore postulate a scheme of type

$$Y_t = \beta_0 + \beta_1 X_{t-1} + \ldots + \beta_s X_{t-s} + u_t \,.$$

We assume that the β_i's can be approximated by a function $f(z)$ i.e. $f(z) \simeq \beta_z$ as below.

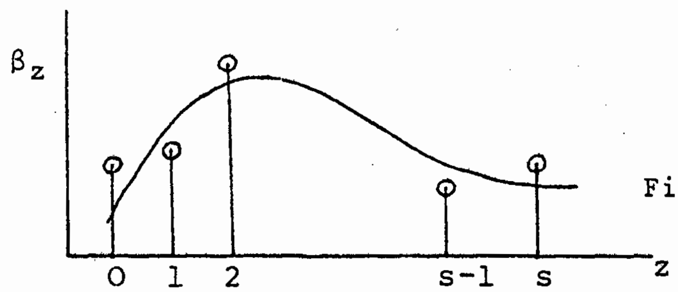

Figure A2

$f(z)$ may be approximated by a polynomial of degree r — clearly when $r = s$ the fit is exact.

$$f(z) \simeq a_0 + a_1 z + \ldots + a_r z^r \qquad \text{(A.4.1)}$$

One of the advantages of the Almon scheme is that it saves degrees of freedom. For example, if $r = 2$ and $s = 5$ we obtain the following scheme for the β's .

$$\beta_0 \triangleq f(0) \triangleq a_0$$

$$\beta_1 \triangleq f(1) \triangleq a_0 + a_1 + a_2 \qquad \text{(A.4.2)}$$

$$\vdots$$

$$\beta_5 \triangleq f(5) \triangleq a_0 + 5a_1 + 25a_2$$

Substitution in our original expression yields

$$\begin{aligned} Y_t &= a_1(X_t + X_{t-1} + \ldots + X_{t-5}) \\ &\quad + a_2(X_{t-1} + 2X_{t-2} + \ldots + 5X_{t-5}) \qquad \text{(A.4.3)} \\ &\quad + a_3(X_{t-1} + 4X_{t-2} + \ldots + 25X_{t-5}) + u_t \ . \end{aligned}$$

We may regress these sets of transformed variables against Y to obtain estimates of a_0, a_1 and a_2 which we can substitute back in (A.4.2) to obtain estimates of the β_i . It is seen that we have in fact saved 3 degrees of freedom by regressing 3 variables against Y in place of the original 6 .

In addition if we assume that the sets of linear combinations in (A.4.3) are less collinear than the original lagged values of X , the problem of multicollinearity will have been reduced.

The AUTO package (Appendix C) contains routines which perform ALMON lag analysis for various r and s .

A.5 SIMULTANEOUS ESTIMATION USING FULL INFORMATION MAXIMUM LIKELIHOOD METHODS INTRODUCTION

Of particular interest in econometric analysis is the concept of a simultaneous linear system of p equations of the form

$$Ay + Bx = u \qquad \text{(A.5.1)}$$

where y is a $(p \times 1)$ vector of endogenous variables

x is a $(q \times 1)$ vector of exogenous and predetermined variables

A is a $(p \times p)$ matrix of the coefficients of the endogenous variables

B is a $q \times p$ matrix of the coefficients of the exogenous and predetermined variables

u is a $(1 \times p)$ vector of disturbances.

If we have T observations on Y and x we may write (A.5.1) as

$$YA + XB = U$$

where Y is a $T \times P$ matrix of T observations, of each of the p jointly dependent variables and X is a $T \times q$ matrix of observations of q predetermined variables. U will be a $T \times p$ matrix of disturbances.

We assume

(i) A is non-singular.

(ii) The rows of U are independently and normally distributed with mean zero and non-singular covariance matrix V.

(iii) The i^{th} endogenous variable in the i^{th} equation is assumed to have coefficient 1.

(iv) Each equation of the system is identified because of a priori conditions on A and B.

*<u>The Full Information Maximum Likelihood Analysis with incorporation of a priori restrictions</u>

Each row of U, U_i; $i = 1,\ldots,T$ has density

$$f(U_i) = \frac{1}{(2\pi)^{\frac{p}{2}}|V|^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}U_i V^{-1} U_i'\right)$$

where $|V|$ denotes the determinant of V.

Under the assumption of row independence

$$f(U_1,U_2,\ldots,U_T) = \prod_{i=1}^{T} f(U_i)$$

$$= (2\pi)^{\frac{-TP}{2}} |V|^{\frac{-T}{2}} \exp -\frac{1}{2} \sum_{i=1}^{T} U_i V^{-1} U_i')$$

*This discussion is taken for the most part from Durbin (unpublished) and used by Wymer (1975) in his estimation procedures.

$$= (2\pi)^{\frac{-TP}{2}} \quad |V|^{\frac{-T}{2}} \exp -\tfrac{1}{2}\mathrm{tr}\ U V^{-1} U'$$

$$= k\ |V|^{\frac{-T}{2}} \exp -\tfrac{1}{2}\mathrm{tr}\ U' U V^{-1}$$

(where k is a constant).

The Jacobian of the transformation $U \rightarrow Y = |A|^T$.

(Anderson (1958))

The likelihood function $f(Y)$ is then

$$L = k|V|^{\frac{-T}{2}} |A|^T \exp -\tfrac{1}{2}\mathrm{tr}\ (YA + XB)'(YA + XB)V^{-1} \ .$$

The log likelihood function

$$\mathrm{Log}\ L = \mathrm{constant} + T \log |A| + \tfrac{1}{2} T \log |V^{-1}| \qquad \text{(A.5.2)}$$
$$- \tfrac{1}{2}\mathrm{tr}\ (YA + XB)'(YA + XB)V^{-1} \ .$$

The maximum likelihood estimates are obtained by equating the derivatives of Log L with respect to A , B and V^{-1} to zero.

We need the following rules for a square matrix A , matrices β and γ (such that $\beta\gamma$ is square) and symmetric matrices δ and ε .

$$\frac{\partial \ \mathrm{Log}\ |A|}{\partial A} = (A')^{-1}$$

$$\frac{\partial \ \mathrm{tr}\ \beta\gamma}{\partial \beta} = \gamma'$$

$$\frac{\partial \ \mathrm{tr}\ \beta'\delta\beta\varepsilon}{\partial \beta} = 2\,\delta\beta\varepsilon$$

We then obtain

$$\frac{\partial \ \mathrm{Log}\ L}{\partial A} = T(A')^{-1} - Y'(YA + XB)V^{-1} \qquad \text{(A.5.3)}$$

$$\frac{\partial \ \mathrm{Log}\ L}{\partial \beta} = X'(YA + XB)V^{-1} \qquad \text{(A.5.4)}$$

$$\frac{\partial \text{ Log } L}{\partial V^{-1}} = \frac{1}{2}TV - \frac{1}{2}(YA + X\ )'(YA + XB) \qquad (A.5.5)$$

Equating (A 5.3,4,5) to zero we obtain:

$$T(A')^{-1} - Y'(YA + XB)V^{-1} = 0 \qquad (A.5.6)$$

$$X'(YA + XB)V^{-1} = 0 \qquad (A.5.7)$$

$$TV - (YA + XB)'(YA + XB) = 0 \qquad (A.5.8)$$

The fact that a priori restrictions have been imposed on A and B (diagonal of 1's in the A matrix and zero elements in the A and B matrices) means that it is only the derivatives with respect to unknown elements which are equated to zero. In (A.5.6) and (A.5.7) therefore (V is unrestricted) it is only functions of the unknown A and B elements that are equated to zero.

Multiplying out (A.5.8) yields

$$TV - A'Y'(YA + XB) = B'X'\ (YA + XB)\ .$$

Premultiplying by $(A')^{-1}$ and post multiplying by V^{-1} we obtain

$$T(A')^{-1} - Y'(YA + XB)V^{-1} + (A')^{-1}B'X'(YA + XB)V^{-1} \qquad (A.5.9)$$

(A 5.6) and (A 5.9) imply $(A')^{-1}B'X'(YA + XB)V^{-1} = 0$ (A.5.10)

We now let W denote the matrix $[Y : X]$ and $\bar{W}$ the matrix $[-XBA^{-1} : X]$ i.e. with Y replaced by the reduced form regression.

Let C denote the matrix $\begin{bmatrix} A \\ \cdot\cdot \\ B \end{bmatrix}$.

Then (A.5.7) and (A.5.10) can be written as

$$\overline{W}'WCV = \begin{bmatrix} (-XBA^{-1})' \\ \cdot\cdot \\ X' \end{bmatrix} [Y : X] \begin{bmatrix} A \\ \cdot\cdot \\ B \end{bmatrix} V^{-1}$$

$$= \begin{bmatrix} (A')^{-1}B'X'(YA + XB)V^{-1} \\ X'(YA + XB)V^{-1} \end{bmatrix}$$

$$= 0 \qquad (A.5.11)$$

where as before only elements corresponding to unknown elements of C are equated to 0 .

Let c_j denote the j^{th} column of C $(j = 1,2,...,p)$. The j^{th} element of c_j is a priori unity and other elements of c_j may be zero. Let m_j equal the number of unknown elements in c_j and let the vector of m_j unknown elements be denoted by $-\delta_j$. Let the columns of W corresponding to unknown elements of c_j be arranged as a $T \times m_j$ matrix Z_j . For example, if the first element of c_j is unknown we make the first column of W equal to the first column of Z_j . We denote the j^{th} column of Y by y_j . A priori Z_j will not include y_j .

In the product Wc_j , y_j has a coefficient of unity and the columns of Z_j have coefficients equal to the elements of $-\delta_j$.

$$\overset{W}{\begin{bmatrix} y_1 \cdots y_j \cdots y_p x_1 \cdots x_q \end{bmatrix}} \overset{c_j}{\begin{bmatrix} A_{1j} \\ \vdots \\ A_{jj}=1 \\ \vdots \\ A_{pj} \\ \vdots \\ B_{1j} \\ \vdots \\ B_{qj} \end{bmatrix}} = y_1A_{1j} + \ldots + y_j + \ldots y_pA_{pj} + x_1B_{1j} + \ldots + x_qB_{qj}$$

i.e. $Wc_j = y_j - Z_j\delta_j$.

Therefore

$$WC = [y_1 - Z_1\delta_1 ,\ldots, y_p - Z_p\delta_p] .$$

Substituting in (A 5.11) we obtain

$$\bar{W}' \sum_{k=1}^{p} (y_k - Z_k\delta_k)[V^{k1},\ldots,V^{kp}] = 0 \qquad \text{(A.5.12)}$$

where V^{kj} is the kj^{th} element of V^{-1} .

To take account of a priori restrictions, we now equate the elements of the left-hand side with unknown elements of C to zero.

The j^{th} column of the left-hand side of (A.5.12) is

$$\bar{W}' \sum_{k=1}^{p} (y_k - Z_k\delta_k)V^{kj} . \qquad \text{(A.5.13)}$$

Let $\bar{Z}_j$ be the $T \times m_j$ matrix formed by the columns of $\bar{W}$ corresponding to unknown elements of c_j i.e. $\bar{Z}_j$ is obtained from $\bar{W}$ by the same process as Z_j is obtained from W . Therefore, $\bar{Z}_j$ is the same as Z_j except that all endogenous variables in Z_j are replaced by their reduced form regression. We wish to set the m_j elements of (A 5.13) corresponding to the m_j unknown elements of c_j to zero.

These m_j elements will by the above be the $m_j \times 1$ vector

$$\bar{Z}_j' \sum_{k=1}^{p} (y_k - Z_k\delta_k)V^{kj} .$$

(A 5.12) is therefore equivalent to

$$\bar{Z}_j' \sum_{k=1}^{p} (y_k - Z_k\delta_k)V^{kj} = 0 \qquad (j = 1,\ldots,p). \qquad \text{(A.5.14)}$$

(A.5.14) may be written as

$$\bar{Z}'Gy = \bar{Z}'GZ\delta \qquad (A.5.15)$$

where

$Z, \bar{Z}$ are $pT \times m$ matrices; $m = \sum_{j=1}^{p} m_j$ equals the total number of regression coefficients to be estimated.

$$Z = \begin{bmatrix} Z_1 & O \cdots\cdots & O \\ O & Z_2 & \vdots \\ \vdots & \ddots & \vdots \\ O & \cdots\cdots & Z_p \end{bmatrix} \qquad \bar{Z} = \begin{bmatrix} \bar{Z}_1 & O \cdots\cdots & O \\ O & \bar{Z}_2 & \vdots \\ \vdots & \ddots & \vdots \\ O & \cdots\cdots & \bar{Z}_p \end{bmatrix}$$

G is the $pT \times pT$ matrix

$$V^{-1} \otimes I_T$$

$$\delta_{(m\times 1)} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_p \end{bmatrix} \qquad Y_{(pT\times 1)} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Denoting the maximum likelihood estimates of $\bar{Z}$, G and δ by $\hat{Z}$, $\hat{G}$ and $\hat{\delta}$ the maximum likelihood estimate of δ will be the solution $\hat{\delta}$ of

$$\hat{Z}'\hat{G}Z\hat{\delta} = \hat{Z}'\hat{G}Y$$

$$\hat{G} = \hat{V}^{-1} \otimes I_T$$

where $\hat{V} = T^{-1}(Y\hat{A} + X\hat{B})'(Y\hat{A} + X\hat{B})$. (A.5.8)

<u>The iterative solution</u>

A form of the Newton-Raphson method can now be applied to (A.5.15) to obtain an iterative procedure for solution.

Assume $\delta(1)$ is an initial estimate of δ .

Let $\delta(2) = \delta(1) + d\delta$ be a second approximation.

Our function is of the form

$$F(\delta) = \overline{Z}'GZ\delta - \overline{Z}'Gy = 0$$
$$= \overline{Z}'G(Z\delta - y)$$

where G and $\overline{Z}$ are functions of δ .

The Taylor series to two terms is

$$F(\delta_{(1)} + d\delta) = F(\delta_{(1)}) + \frac{dF(\delta_{(1)})}{1!}$$
$$= Z'_{(1)}\, G_{(1)}\, (Z\delta_{(1)} - y)$$
$$+ d(\overline{Z}'G).(Z\delta - y)\Big|_{(\delta_{(1)})} + \overline{Z}'G(d(Z\delta - y))\Big|_{(\delta_{(1)})}$$

$$= \overline{Z}'_{(1)}G_{(1)}(Z\delta_{(1)} - y) + (dZ'G_{(1)} + \overline{Z}'_{(1)}dG)(Z\delta_{(1)} - y) + \overline{Z}'_{(1)}G_{(1)}(Zd\delta)$$

(Z, y are constants).

$\overline{Z}_{(1)}$, $G_{(1)}$ are the values of $\overline{Z}$, G calculated at $\delta_{(1)}$. and $d\overline{Z}$, dG are the increments in $\overline{Z}$ and G due to the change from $\delta_{(1)}$ to $\delta_{(1)}$. Since the elements of $d\overline{Z}'G_{(1)}$ and $\overline{Z}_{(1)}dG$ are small compared with those of $\overline{Z}'_{(1)}\, G_{(1)}$ we can write

$$d\delta = (\overline{Z}'_{(1)}\, G_{(1)}\, Z)^{-1}\, \overline{Z}_{(1)}\, G_{(1)}\, (y - Z\delta_{(1)})$$

i.e. $\delta(z) = (\overline{Z}'_{(1)}\, G_{(1)}\, Z)^{-1}\, \overline{Z}'_{(1)}\, G_{(1)}\, y$.

Inductively, we establish the following general formula for the $(r + 1)^{th}$ iteration

$$\delta_{(r+1)} = \delta_{(r)} + (\overline{Z}'_{(r)}\, G_{(r)}\, Z)\ \ \overline{Z}'_{(r)}G_{(r)}(y - Z\delta_{(r)}) .$$

A.6 *Asymptotic Solution paths of differential equations of the form $Dx = Ax$; A a real matrix.

We consider for simplicity A a 2×2 matrix.

x is therefore of the form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

<u>The following special cases apply</u>

<u>Case I</u> - A has real eigenvalues of opposite signs. Since A has distinct eigenvalues we know (Bellman (1970), p. 194) that A is diagonalizable.

$$\text{i.e. } \exists \; P \;\; \text{s.t.} \;\; B = PAP^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

$$\lambda < 0 < \mu \; ;$$

If we change coordinates to $y = Px$; $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we obtain

$$Dy = PDx = PAx = PA(P^{-1}y)$$

$$\text{i.e. } Dy = By \tag{A.6.1}$$

(A.6.1) has solutions $y_1(t) = y_1(0) \exp(\lambda t)$

$$y_2(t) = y_2(0) \exp(\mu t)$$

$$\therefore \quad x(t) = P^{-1}\begin{pmatrix} y_1(0) \exp(\lambda t) \\ y_2(0) \exp(\mu t) \end{pmatrix}$$

$$= P^{-1}\begin{pmatrix} \exp \lambda t & 0 \\ 0 & \exp \mu t \end{pmatrix} y(0)$$

$$= P^{-1}\begin{pmatrix} \exp \lambda t & 0 \\ 0 & \exp \mu t \end{pmatrix} Px(0)$$

* This discussion is taken largely from Hirsch and Smale (1974).

A.6.2 PHASE DIAGRAM REPRESENTATION

We may plot the solution paths $x(t)$ of the differential equation for increasing t and variable initial solution $x(0)$ in the form of a phase diagram Hirsch and Smale (1974) .

For the above case we obtain

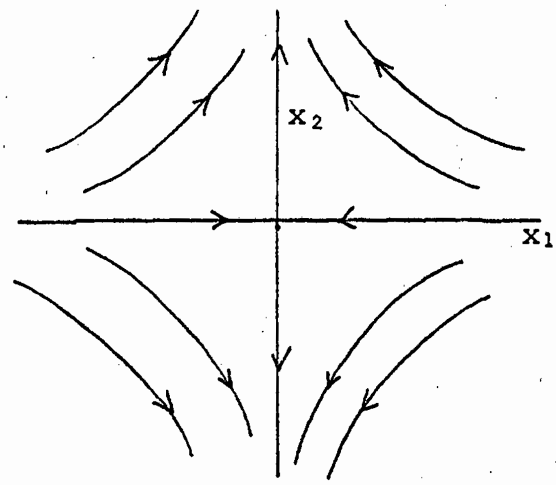

Figure A3

Case II - A has eigenvalues with negative real parts.

(i) If the eigenvalues are distinct and A is therefore diagonalizable we use the above transformation i.e.

$$B = PAP^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad 0 < \lambda \neq \mu < 0$$

to obtain the solution (by the above methods)

$$x(t) = P^{-1}\begin{bmatrix} \exp \lambda t & 0 \\ 0 & \exp \mu t \end{bmatrix} Px(0)$$

(ii) If the eigenvalues are not distince i.e. A not diagonalizable we transform by

$$B = PAP^{-1} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \qquad \lambda < 0$$

to obtain

$$x(t) = P^{-1}\begin{bmatrix} e^{\lambda t} & 0 \\ te^{t\lambda} & e^{t\lambda}\end{bmatrix}Px(0)$$

(iii) If the eigenvalues are complex with negative real parts i.e. of the form $a \pm ib$ with $a < 0$ we transform by

$$B = PAP^{-1} = \begin{bmatrix} a & -b \\ b & a\end{bmatrix}$$

to obtain

$$x(t) = P^{-1}\begin{bmatrix} e^{ta}\cos tb & -e^{ta}\sin tb \\ e^{ta}\sin tb & -e^{ta}\cos tb\end{bmatrix}Px(0)$$

also $x(t) = x\left(t + \frac{2\pi}{b}\right)$ so this solution is periodic with period $\frac{2\pi}{b}$.

In (i) and (ii) it is clear that $\lim_{t\to\infty} x(t) = 0$.

In case (iii) we note that since

$$|\cos tb| \leqslant 1, \quad |\sin tb| \leqslant 1 \quad \text{and} \quad a < 0, \quad \lim_{t\to\infty} x(t) = 0.$$

Phase diagram representations are to be found in Hirsch and Smale (Chapter 5).

Case III

All eigenvalues have positive real parts. The solutions are as in Case II except that $\lambda, \mu, a > 0$. It is clear therefore that

$$\lim_{t\to\infty} x(t) = \infty .$$

Phrase diagram representation as in Case II but with arrows reversed.

APPENDIX B

THE DATA

Apart from the series on export prices*, all the following data series have been extracted from the South African Reserve Bank Quarterly Bulletins, and their supplements.

(a) Data Series used in Chapter Four for annual estimation over the period 1946-1975. All figures at 1963 constant prices.

1. C = Private consumption expenditure
2. Y = Net domestic product
3. K = Fixed capital stock
4. S = Inventory level
5. T = Direct taxes minus transfers from government and rest of world
6. G = Consumption expenditure by general government
7. I = Imports of goods and non-factor services
8. E = Exports of goods and non-factor services

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* Taken from the "Bulletin of Statistics."
Published by the Department of Statistics, Pretoria

	1	2	3	4	5	6	7	8
1	2179.00	2905.00	5632.00	556.00	250.00	288.00	1022.00	663.00
2	2319.00	3059.00	6076.00	755.00	292.00	307.00	1321.00	677.00
3	2461.00	3321.00	6621.00	1033.00	250.00	297.00	1379.00	782.00
4	2523.00	3401.00	7163.00	1148.00	274.00	293.00	1132.00	765.00
5	2576.00	3594.00	7611.00	1150.00	297.00	349.00	997.00	794.00
6	2691.00	3801.00	8016.00	1122.00	321.00	343.00	1244.00	841.00
7	2753.00	3904.00	8512.00	1264.00	368.00	398.00	1036.00	906.00
8	2861.00	4107.00	9094.00	1101.00	410.00	428.00	1099.00	940.00
9	2949.00	4381.00	9672.00	1042.00	393.00	413.00	1120.00	1113.00
10	3071.00	4620.00	10165.00	1063.00	379.00	436.00	1217.00	1239.00
11	3161.00	4861.00	10656.00	1140.00	431.00	463.00	1237.00	1325.00
12	3302.00	5076.00	11161.00	1240.00	440.00	467.00	1367.00	1423.00
13	3385.00	5189.00	11774.00	1372.00	422.00	477.00	1389.00	1387.00
14	3456.00	5369.00	12315.00	1451.00	439.00	482.00	1223.00	1595.00
15	3556.00	5551.00	12828.00	1422.00	446.00	516.00	1354.00	1649.00
16	3607.00	5735.00	13327.00	1521.00	461.00	556.00	1232.00	1744.00
17	3733.00	6054.00	13775.00	1584.00	458.00	664.00	1249.00	1801.00
18	3974.00	6547.00	14392.00	1651.00	659.00	690.00	1518.00	1921.00
19	4335.00	6986.00	15228.00	1854.00	662.00	749.00	1827.00	2007.00
20	4548.00	7448.00	16274.00	2001.00	729.00	819.00	1997.00	2025.00
21	4748.00	7799.00	17236.00	2206.00	736.00	853.00	1789.00	2160.00
22	4966.00	8391.00	18163.00	2232.00	884.00	882.00	2130.00	2354.00
23	5331.00	8712.00	19098.00	2586.00	951.00	935.00	2147.00	2565.00
24	5701.00	9325.00	20145.00	2712.00	1012.00	1010.00	2369.00	2514.00
25	6109.00	9797.00	21335.00	2983.00	1071.00	1101.00	2757.00	2514.00
26	6303.00	10266.00	22723.00	3332.00	1095.00	1183.00	3015.00	2671.00
27	6444.00	10575.00	24125.00	3677.00	1191.00	1215.00	2592.00	3043.00
28	6801.00	10951.00	25755.00	3360.00	1441.00	1244.00	2939.00	2990.00
29	7148.00	11725.00	27509.00	3383.00	1712.00	1383.00	3579.00	2938.00
30	7369.00	12004.00	29341.00	3805.00	1751.00	1575.00	3443.00	2961.00

(b) Data Series set up for use in Chapters Two and Four for quarterly estimation over the period 1960-1974 (at current prices).

1.	Gde	= Gross domestic expenditure
2.		= Private consumption expenditure on durable and semi durables
3.		= Private consumption expenditure on non durables and services
2. + 3.		= C
4.	ΔS	= Change in inventory level
5.	G	= Consumption expenditure by general government
6.		= Gross domestic product (at factor cost) in agricultural sector
7.	GNA	= Gross domestic product (at factor cost) in non-agricultural sector
8.	Y	= Gross domestic product
9.	$^{+}D_{s}Y$ or $^{\dagger}DY$	= Disposable Income
10.	Imp	= Imports of goods and non-factor services
11.	Exp	= Exports of goods and non-factor services
12.	P_{Imp}	= Price index of imported goods (1970 = 100)
13.	$^{+}P_{Exp}$ or $^{\dagger}P_{E}$	= Price index of exported goods (1970 = 100)
14.	$P_{e}TAX$	= Personal direct taxes
15.		= Company Tax
16.		= Indirect Tax
17.		= Company Saving
18.		= Subsidies
19.	LNA	= Labour in non-agricultural sector
20.	M_{s}	= Money Supply
21.	$^{\dagger}r$ or $^{+}i_{ST}$	= Short term interest rate (treasury bill rate)
22.	i_{LT}	= Long term interest rate (rate on long term government stock)

23.		= Long term interest rate (rate on long term government stock)
24.		= Income from property by government less interest on public debt
25.		= Current transfers received from general government plus the rest of the world
26.	W	= Wages (renumeration of employees)
27.		= Profits (gross operating surplus)
28.		= Gross investment in the Agricultural Sector
29.		= Depreciation in Agricultural Sector
30.	P	= Wholesale Price Index (1970 = 100)
31.	I	= Gross Investment
32.		= Depreciation
33.	K	= Real capital stock*
34.	KNA	= Real capital stock in non-agricultural sector*
35.	S	= Inventory level*

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* at 1970 prices

† in Chapter Two

+ in Chapter Four

1	1177.00	210.00	535.00	1.00	125.00	137.00	1037.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	1217.00	225.00	658.00	23.00	111.00	143.00	1045.00	1287.00	916.00	395.00	415.00	83.40	94.00	25.00	12.00	87.00
3	1284.00	217.00	610.00	40.00	127.00	156.00	1119.00	1350.00	975.00	333.00	399.00	83.60	94.00	19.00	27.00	86.00
4	1317.00	268.00	641.00	20.00	122.00	165.00	1143.00	1387.00	930.00	311.00	381.00	84.00	94.00	80.00	75.00	91.00
5	1245.00	223.00	620.00	-26.00	133.00	131.00	1115.00	1335.00	940.00	325.00	415.00	84.10	94.00	66.00	124.00	98.00
6	1240.00	224.00	644.00	29.00	114.00	150.00	1112.00	1339.00	960.00	319.00	418.00	84.10	93.00	27.00	14.00	85.00
7	1303.00	211.00	634.00	105.00	143.00	207.00	1162.00	1442.00	1157.00	272.00	411.00	84.60	95.00	23.00	34.00	83.00
8	1283.00	256.00	671.00	-29.00	135.00	176.00	1203.00	1452.00	1010.00	279.00	440.00	84.90	94.00	82.00	82.00	86.00
9	1270.00	216.00	644.00	-57.00	162.00	140.00	1189.00	1416.00	992.00	290.00	436.00	85.10	94.00	71.00	121.00	97.00
10	1262.00	231.00	646.00	-2.00	131.00	165.00	1197.00	1442.00	1054.00	295.00	475.00	85.50	94.00	28.00	15.00	90.00
11	1397.00	230.00	647.00	106.00	155.00	203.00	1243.00	1529.00	1188.00	305.00	437.00	86.00	94.00	20.00	35.00	95.00
12	1454.00	293.00	716.00	1.00	171.00	174.00	1291.00	1550.00	1078.00	339.00	435.00	86.50	96.00	86.00	78.00	101.00
13	1444.00	239.00	681.00	-19.00	187.00	155.00	1276.00	1533.00	1034.00	365.00	454.00	86.90	94.00	90.00	153.00	112.00
14	1489.00	262.00	716.00	41.00	146.00	178.00	1333.00	1600.00	1088.00	379.00	490.00	87.10	95.00	56.00	66.00	102.00
15	1607.00	255.00	717.00	166.00	187.00	219.00	1403.00	1712.00	1302.00	374.00	479.00	87.10	93.00	61.00	86.00	103.00
16	1659.00	333.00	746.00	23.00	170.00	201.00	1459.00	1756.00	1194.00	400.00	498.00	87.90	95.00	68.00	85.00	111.00
17	1653.00	274.00	747.00	-29.00	218.00	179.00	1425.00	1720.00	1142.00	432.00	499.00	88.50	97.00	95.00	105.00	129.00
18	1707.00	303.00	686.00	77.00	170.00	195.00	1485.00	1783.00	1247.00	450.00	526.00	89.10	98.00	65.00	68.00	116.00
19	1733.00	307.00	691.00	59.00	200.00	159.00	1554.00	1819.00	1241.00	471.00	507.00	89.80	97.00	73.00	107.00	120.00
20	1903.00	383.00	840.00	40.00	191.00	178.00	1640.00	1931.00	1296.00	487.00	515.00	90.30	97.00	75.00	101.00	131.00
21	1914.00	309.00	800.00	27.00	217.00	187.00	1600.00	1912.00	1244.00	508.00	506.00	90.50	99.00	117.00	114.00	140.00
22	2039.00	329.00	853.00	96.00	197.00	222.00	1642.00	1974.00	1274.00	584.00	519.00	90.80	99.00	67.00	85.00	124.00
23	2014.00	312.00	856.00	177.00	231.00	173.00	1693.00	1973.00	1418.00	544.00	503.00	91.20	99.00	101.00	123.00	124.00
24	1972.00	398.00	907.00	-29.00	223.00	178.00	1779.00	2064.00	1489.00	471.00	563.00	91.70	97.00	77.00	103.00	130.00
25	1939.00	331.00	845.00	-37.00	269.00	170.00	1723.00	2017.00	1396.00	456.00	536.00	92.60	98.00	134.00	142.00	140.00
26	2026.00	355.00	921.00	42.00	228.00	240.00	1770.00	2126.00	1533.00	450.00	550.00	93.60	103.00	80.00	79.00	129.00
27	2097.00	348.00	940.00	28.00	247.00	211.00	1853.00	2186.00	1477.00	521.00	610.00	94.30	101.00	123.00	133.00	141.00
28	2238.00	442.00	972.00	11.00	234.00	209.00	1925.00	2262.00	1578.00	541.00	545.00	94.50	101.00	88.00	102.00	152.00
29	2240.00	360.00	944.00	97.00	280.00	173.00	1898.00	2218.00	1410.00	599.00	577.00	94.80	101.00	164.00	167.00	169.00
30	2351.00	394.00	1001.00	209.00	225.00	254.00	1943.00	2326.00	1704.00	615.00	590.00	95.40	102.00	85.00	106.00	157.00
31	2441.00	383.00	959.00	249.00	282.00	349.00	2000.00	2501.00	1752.00	579.00	639.00	95.50	97.00	139.00	158.00	156.00
32	2364.00	469.00	1049.00	35.00	254.00	246.00	2098.00	2478.00	1819.00	514.00	628.00	95.90	98.00	94.00	114.00	162.00
33	2300.00	388.00	1023.00	-15.00	303.00	210.00	2055.00	2428.00	1562.00	544.00	672.00	96.00	96.00	180.00	181.00	186.00
34	2424.00	434.00	1039.00	40.00	248.00	251.00	2128.00	2520.00	1832.00	567.00	663.00	96.20	98.00	90.00	104.00	164.00
35	2489.00	436.00	1095.00	70.00	306.00	244.00	2217.00	2609.00	1748.00	571.00	691.00	96.30	97.00	154.00	164.00	173.00
36	2642.00	542.00	1159.00	47.00	293.00	244.00	2294.00	2698.00	1949.00	586.00	642.00	96.70	99.00	110.00	142.00	191.00
37	2483.00	456.00	1103.00	-90.00	343.00	203.00	2271.00	2652.00	1717.00	549.00	718.00	97.00	97.00	197.00	197.00	212.00
38	2857.00	506.00	1190.00	193.00	285.00	291.00	2392.00	2870.00	2041.00	637.00	650.00	97.60	98.00	112.00	140.00	212.00
39	3031.00	509.00	1207.00	229.00	328.00	275.00	2509.00	3003.00	1932.00	692.00	664.00	98.00	99.00	138.00	202.00	246.00
40	3025.00	643.00	1279.00	41.00	357.00	223.00	2571.00	3010.00	2101.00	714.00	699.00	98.50	103.00	92.00	161.00	247.00
41	2939.00	523.00	1235.00	56.00	396.00	195.00	2529.00	2952.00	1920.00	683.00	646.00	99.20	103.00	178.00	253.00	261.00
42	3211.00	607.00	1311.00	235.00	341.00	320.00	2631.00	3160.00	2359.00	768.00	717.00	99.30	100.00	104.00	154.00	240.00
43	3233.00	601.00	1351.00	119.00	417.00	219.00	2719.00	3152.00	2124.00	812.00	681.00	100.90	98.00	163.00	231.00	255.00
44	3484.00	732.00	1427.00	72.00	398.00	238.00	2840.00	3312.00	2388.00	848.00	676.00	101.50	101.00	120.00	160.00	276.00
45	3471.00	581.00	1377.00	58.00	486.00	195.00	2775.00	3227.00	2148.00	901.00	657.00	102.90	100.00	216.00	239.00	299.00
46	3565.00	641.00	1456.00	106.00	406.00	370.00	2862.00	3454.00	2446.00	874.00	763.00	104.70	101.00	126.00	127.00	260.00
47	3724.00	648.00	1484.00	242.00	494.00	337.00	3010.00	3595.00	2520.00	896.00	768.00	105.70	100.00	213.00	241.00	296.00
48	3754.00	797.00	1610.00	15.00	476.00	241.00	3151.00	3674.00	2774.00	876.00	796.00	107.40	104.00	148.00	171.00	309.00
49	3620.00	632.00	1523.00	-143.00	519.00	218.00	3127.00	3642.00	2422.00	849.00	871.00	113.90	110.00	256.00	250.00	340.00
50	3636.00	693.00	1585.00	-46.00	447.00	369.00	3174.00	3779.00	2819.00	838.00	981.00	117.00	119.00	154.00	154.00	278.00
51	3775.00	706.00	1656.00	11.00	526.00	388.00	3352.00	3995.00	2860.00	845.00	1065.00	119.80	122.00	236.00	290.00	297.00
52	4100.00	850.00	1756.00	-155.00	485.00	341.00	3575.00	4196.00	2922.00	931.00	1027.00	123.50	124.00	165.00	215.00	327.00
53	3997.00	714.00	1742.00	-142.00	553.00	292.00	3610.00	4205.00	2622.00	1001.00	1209.00	125.90	134.00	352.00	311.00	353.00
54	4394.00	814.00	1841.00	21.00	486.00	382.00	3951.00	4622.00	3145.00	1009.00	1237.00	129.50	151.00	158.00	213.00	323.00
55	4690.00	838.00	1845.00	44.00	632.00	421.00	4253.00	4982.00	3093.00	1103.00	1395.00	134.20	166.00	322.00	387.00	350.00
56	5218.00	1021.00	2074.00	103.00	590.00	348.00	4507.00	5201.00	3335.00	1176.00	1159.00	139.60	166.00	498.00	329.00	390.00
57	5073.00	829.00	2020.00	81.00	722.00	318.00	4517.00	5194.00	3125.00	1334.00	1455.00	146.20	171.00	412.00	461.00	410.00
58	5511.00	945.00	2172.00	402.00	587.00	610.00	4629.00	5618.00	3817.00	1654.00	1761.00	156.60	224.00	195.00	355.00	377.00
59	5901.00	951.00	2249.00	361.00	776.00	594.00	4854.00	5786.00	3737.00	1819.00	1704.00	173.30	220.00	394.00	586.00	389.00
60	6188.00	1147.00	2494.00	129.00	755.00	371.00	5306.00	6014.00	3939.00	1895.00	1711.00	178.20	229.00	245.00	357.00	405.00

1	-73.00	9.0027211.87	1311.00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	117.00	8.0027129.99	1329.00	3.60	5.25	66.00	5.00	39.00	681.00	527.00	26.00	12.00	80.00	256.00	117.00
3	71.00	11.0026710.49	1343.00	3.60	5.37	65.00	15.00	40.00	700.00	575.00	32.00	13.00	80.80	267.00	120.00
4	79.00	12.0026710.49	1347.00	3.90	5.37	59.00	18.00	43.00	715.00	593.00	31.00	13.00	81.10	273.00	122.00
5	-90.00	9.0026960.13	1305.00	4.22	5.62	72.00	14.00	38.00	711.00	535.00	24.00	13.00	81.70	260.00	123.00
6	139.00	8.0026718.41	1360.00	4.60	5.62	74.00	3.00	40.00	720.00	542.00	26.00	13.00	81.40	272.00	125.00
7	45.00	10.0026976.23	1414.00	4.20	5.87	69.00	12.00	44.00	743.00	626.00	31.00	13.00	81.80	267.00	127.00
8	71.00	13.0027046.07	1466.00	4.03	5.87	54.00	14.00	45.00	750.00	629.00	29.00	13.00	81.70	269.00	130.00
9	-74.00	10.0027074.04	1452.00	3.60	5.87	67.00	19.00	46.00	750.00	579.00	29.00	13.00	81.80	253.00	131.00
10	140.00	10.0027353.73	1557.00	2.65	5.62	61.00	1.00	44.00	761.00	601.00	27.00	13.00	82.40	262.00	133.00
11	67.00	12.0027689.36	1623.00	2.21	5.00	64.00	17.00	48.00	787.00	659.00	33.00	14.00	82.30	271.00	135.00
12	72.00	16.0027657.17	1659.00	1.85	4.75	60.00	16.00	49.00	806.00	659.00	31.00	14.00	83.20	286.00	137.00
13	-83.00	10.0027445.61	1652.00	2.03	4.75	69.00	17.00	46.00	818.00	613.00	24.00	14.00	83.10	297.00	139.00
14	132.00	13.0027717.33	1726.00	2.17	4.75	66.00	5.00	48.00	838.00	673.00	29.00	14.00	83.50	320.00	142.00
15	46.00	13.0028164.83	1783.00	1.85	4.75	61.00	24.00	54.00	873.00	749.00	36.00	14.00	83.70	331.00	145.00
16	122.00	15.0028528.43	1846.00	1.96	4.75	54.00	29.00	52.00	887.00	773.00	35.00	15.00	83.70	354.00	148.00
17	-17.00	13.0028724.21	1822.00	2.63	4.75	66.00	29.00	43.00	895.00	709.00	27.00	15.00	84.40	357.00	151.00
18	130.00	14.0029199.69	1934.00	2.86	4.75	76.00	12.00	53.00	918.00	763.00	31.00	15.00	84.90	390.00	155.00
19	99.00	14.0029675.16	1970.00	3.13	4.75	65.00	25.00	54.00	960.00	753.00	40.00	15.00	85.90	428.00	159.00
20	130.00	18.0029731.10	2108.00	3.61	5.00	66.00	30.00	56.00	1001.00	817.00	38.00	15.00	86.90	436.00	163.00
21	3.00	15.0030626.11	2107.00	4.11	5.25	77.00	16.00	60.00	1004.00	783.00	30.00	15.00	87.50	452.00	167.00
22	133.00	14.0031073.61	2179.00	3.98	5.50	81.00	9.00	47.00	1026.00	838.00	33.00	16.00	87.90	473.00	171.00
23	86.00	17.0031521.12	2243.00	4.24	6.00	68.00	19.00	62.00	1067.00	799.00	39.00	16.00	88.30	500.00	175.00
24	94.00	23.0031940.66	2357.00	4.26	6.00	77.00	27.00	62.00	1119.00	838.00	35.00	16.00	89.10	552.00	181.00
25	22.00	16.0032220.35	2488.00	4.34	6.00	75.00	-10.00	58.00	1136.00	757.00	28.00	17.00	89.90	503.00	184.00
26	125.00	21.0032276.28	2372.00	3.84	6.00	79.00	-6.00	60.00	1141.00	877.00	34.00	17.00	91.00	480.00	188.00
27	109.00	19.0032416.13	2596.00	4.25	6.50	79.00	7.00	70.00	1178.00	886.00	39.00	18.00	92.60	521.00	193.00
28	79.00	24.0032619.88	2694.00	4.56	6.50	91.00	29.00	62.00	1208.00	926.00	37.00	18.00	92.90	548.00	198.00
29	51.00	22.0032779.73	2559.00	5.00	6.50	86.00	22.00	66.00	1204.00	867.00	30.00	17.00	93.20	524.00	202.00
30	68.00	28.0033003.48	2682.00	4.94	6.50	86.00	14.00	62.00	1227.00	976.00	37.00	17.00	94.00	533.00	207.00
31	117.00	24.0033087.39	2675.00	4.82	6.50	78.00	31.00	73.00	1275.00	1094.00	47.00	18.00	94.00	573.00	212.00
32	65.00	28.0033395.05	2857.00	5.01	6.50	99.00	28.00	70.00	1322.00	1022.00	38.00	18.00	94.00	579.00	217.00
33	60.00	23.0033730.67	2828.00	5.08	6.50	86.00	13.00	77.00	1319.00	946.00	37.00	19.00	94.30	564.00	223.00
34	73.00	23.0033814.58	3037.00	4.78	6.50	90.00	.00	75.00	1355.00	1024.00	43.00	19.00	94.60	575.00	227.00
35	156.00	25.0034206.15	3199.00	4.50	6.50	96.00	-8.00	86.00	1420.00	1041.00	55.00	20.00	94.80	578.00	233.00
36	44.00	31.0034513.81	3448.00	4.66	6.50	122.00	34.00	77.00	1461.00	1077.00	60.00	20.00	95.80	624.00	236.00
37	70.00	34.0034933.34	3379.00	4.68	6.50	108.00	-16.00	89.00	1452.00	1022.00	41.00	20.00	96.20	623.00	242.00
38	94.00	25.0035408.82	3468.00	4.69	6.50	99.00	1.00	84.00	1483.00	1200.00	49.00	20.00	96.60	651.00	248.00
39	195.00	27.0035688.51	3590.00	4.60	6.50	95.00	-26.00	92.00	1558.00	1226.00	58.00	21.00	97.20	672.00	254.00
40	111.00	31.0035976.17	3775.00	4.23	6.50	143.00	30.00	86.00	1610.00	1184.00	54.00	21.00	98.10	724.00	261.00
41	20.00	33.0036359.76	3701.00	4.44	6.50	108.00	5.00	95.00	1625.00	1099.00	45.00	21.00	98.80	711.00	267.00
42	106.00	31.0036975.08	3902.00	4.36	7.00	115.00	6.00	80.00	1673.00	1278.00	54.00	21.00	99.60	805.00	275.00
43	175.00	41.0037542.43	3933.00	4.38	7.75	116.00	-19.00	98.00	1769.00	1169.00	63.00	22.00	101.10	833.00	283.00
44	89.30	42.0037931.97	3983.00	4.53	7.75	146.00	11.00	91.00	1860.00	1218.00	56.00	23.00	101.90	871.00	291.00
45	-48.00	42.0038093.85	3902.00	5.04	8.50	128.00	-19.00	95.00	1882.00	1088.00	49.00	23.00	102.70	872.00	304.00
46	139.00	38.0038233.69	4034.00	5.51	8.50	117.00	3.00	89.00	1901.00	1331.00	60.00	24.00	104.80	906.00	313.00
47	174.00	47.0038287.63	4151.00	5.54	8.50	108.00	-15.00	116.00	2011.00	1336.00	74.00	24.00	106.00	955.00	321.00
48	182.00	47.0038709.17	4276.00	5.96	8.50	127.00	6.00	106.00	2086.00	1326.00	65.00	25.00	106.90	1056.00	330.00
49	44.00	43.0038625.26	4105.00	5.94	8.50	112.00	-31.00	114.00	2065.00	1280.00	56.00	25.00	110.20	1022.00	339.00
50	140.00	42.0038709.17	4407.00	5.53	8.50	118.00	-24.00	108.00	2100.00	1443.00	58.00	25.00	112.50	1015.00	348.00
51	225.00	42.0038940.89	4695.00	4.96	8.25	147.00	-24.00	135.00	2198.00	1542.00	77.00	26.00	115.70	1092.00	357.00
52	223.00	47.0039520.27	4863.00	4.38	8.12	177.00	-20.00	106.00	2274.00	1642.00	71.00	26.00	118.70	1180.00	366.00
53	163.00	50.0039939.80	5062.00	4.19	7.75	150.00	24.00	126.00	2391.00	1561.00	60.00	26.00	124.10	1102.00	378.00
54	355.00	34.0040331.37	5446.00	2.64	7.75	168.00	-28.00	118.00	2438.00	1895.00	69.00	27.00	127.10	1180.00	388.00
55	368.00	42.0040806.20	5748.00	2.65	7.75	170.00	17.00	142.00	2597.00	2077.00	79.00	28.00	131.40	1252.00	400.00
56	466.00	44.0041141.90	5983.00	3.28	8.00	142.00	47.00	130.00	2690.00	2165.00	74.00	29.00	136.50	1356.00	412.00
57	288.00	51.0041490.30	6161.00	4.82	8.25	195.00	-2.00	148.00	2706.00	2129.00	66.00	29.00	141.40	1342.00	425.00
58	447.00	48.0041701.40	6487.00	5.72	8.50	229.00	-24.00	160.00	2814.00	2475.00	72.00	29.00	150.90	1414.00	439.00
59	160.00	51.0041939.80	6657.00	6.13	9.75	225.00	52.00	174.00	3102.00	2346.00	83.00	30.00	158.30	1543.00	454.00
60	474.00	68.0042074.40	7317.00	5.92	9.50	234.00	71.00	175.00	3260.00	2417.00	90.00	30.00	163.00	1640.00	469.00

33 34	3
115406.0014184.00	1812.00
216059.7514390.25	1847.00
316241.6814498.67	1696.50
416427.8714662.66	1921.17
516575.5614816.88	1889.34
616776.1514981.50	1924.97
716947.3015130.65	2053.33
817117.4315281.20	2017.83
917266.5715416.89	1945.71
1017423.1315556.46	1953.29
1117580.3815698.62	2072.08
1217767.4615857.27	2073.28
1317957.5916035.37	2050.42
1418170.7716230.58	2097.52
1518392.9916426.52	2297.84
1618639.1116648.74	2325.32
1718883.1816878.60	2290.96
1819159.9817136.55	2381.66
1919473.1317420.60	2450.34
2019787.2917708.29	2436.37
2120113.0018016.86	2527.23
2220456.5718341.09	2636.44
2320824.6418683.11	2836.90
2421241.0219078.17	2748.23
2521595.8619420.77	2707.08
2621916.7419722.97	2753.23
2722270.9520054.50	2733.47
2822647.7020410.80	2795.31
2922993.2020742.34	2899.39
3023340.0021067.88	3121.73
3123724.0521421.07	3386.62
3224109.1521784.90	3423.85
3324470.7622127.42	3407.95
3424838.6322469.91	3450.23
3525202.5522796.92	3524.07
3625607.5623170.61	3573.13
3726003.6123544.83	3479.58
3826420.8023932.00	3677.37
3926850.8424323.97	3919.97
4027322.8024762.30	3956.76
4127772.2025187.40	4013.44
4228304.3325686.40	4249.38
4328848.3426189.86	4346.10
4429417.5326726.66	4436.76
4529970.5927254.41	4493.23
4630536.4327785.90	4594.38
4731134.5528336.84	4822.68
4831813.6928978.56	4836.71
4932433.4729570.22	4706.95
5033026.3630133.77	4666.06
5133661.6230724.95	4675.57
5234347.3831372.81	4544.98
5334930.7831928.81	4430.56
5435553.9132518.90	4447.08
5536202.3133128.48	4480.57
5636893.3933787.09	4556.03
5737542.4034409.44	4613.31
5838188.5335027.07	4879.71
5938876.4635681.52	5101.44
6039594.8736363.11	5130.50

A P P E N D I X C

COMPUTER PROGRAMS

The program ECON, written by M.R. Norman of the Wharton School of Finance and Commerce, University of Pennsylvania, was used to obtain the O.L.S. estimates of Chapter Two. In addition AUTO, also written by M.R. Norman, was used to obtain Cochrane-Orcutt estimatesof the parameters. A program written by the author was used to obtain the Ridge and characteristic root analysis of Chapter Two.

The analysis of Chapter Four was carried out using the programs TRANSF, RESIMUL and CONTINEST written by C.R. Wymer of the London School of Economics.

The Almon Lag estimation of Chapter Five was performed using the package AUTO.

A listing is given below of the Ridge/characteristic-root program plus the TRANSF, RESIMUL and CONTINEST input used in Chapter Four for the log and log-linear models using quarterly data 1960-1974.

```
BARN*RIDGE(1).INU
      DIMENSION X(90,90),Y(90),BETA(50),XTX(50,50),XTY(50),SUMSQX(50),ST
     1O(50,50),S1(200),S2(102,102),RSTO(50)
      REAL JC(50),V(2),M(50)
      REAL Z(50,50),D(50),B(50),EIGNVL(50),T1(50),T2(50),EIGNVC(50,50)
      REAL PVL(50)
      REAL  AC(30),XPR(61,61),PR(80),ER(80)
      REAL BETAC(50)
      READ(8,2)J)(ICOL,IROW,NIT,NIT,ILAM,IGAM)
      DO 999 I=1,IROW
  999 READ(8,210)(Y(I),(X(I,J),J=1,ICOL))
  210 FORMAT()
      ICOLX=ICOL+1
      COLX=FLOAT(ICOLX)
      ROWS=FLOAT(IROW)
      SUMY=0.
      SUMSQY=0.
      MCOL=50
      MROW=50
      WRITE(5,91)
      DO 5 I=1,IROW
    5 AC(I)=Y(I)
      DO  6 I=1,IROW
      DO  6 J=1,ICOL
    6 XPR(I,J)=X(I,J)
      DO 23 I=1,IROW
   23 WRITE(5,101)(Y(I),(X(I,J),J=1,ICOL))
C
C     STANDARDIZATION SUBROUTIUE
C
      DO 51 I=1,IROW
      SUMY=SUMY+Y(I)
   51 SUMSQY=SUMSQY+Y(I)*Y(I)
      SUMSQY=SQRT(SUMSQY-SUMY*SUMY/ROWS)
      YBAR=SUMY/ROWS
      T=SUMY
      TSS=SUMSQY*SUMSQY
      DO 33 I=1,ICOL
      RSTO(I)=0
      DO 33 J=1,IROW
   33 RSTO(I)=RSTO(I)+Y(J)*X(J,I)
      DO 52 I=1,IROW
   52 Y(I)=(Y(I)-YBAR)/SUMSQY
      DO 53 I=1,ICOL
      M(I)=0
      SUMSQX(I)=0.
      DO 54 J=1,IROW
      M(I)=M(I)+X(J,I)
   54 SUMSQX(I)=SUMSQX(I)+X(J,I)**2
      SUMSQX(I)=SQRT(SUMSQX(I)-M(I)*M(I)/ROWS)
      M(I)=M(I)/ROWS
      DO 55 J=1,IROW
   55 X(J,I)=(X(J,I)-M(I))/SUMSQX(I)
   53 CONTINUE
      WRITE(5,93)
      DO 24 I=1,IROW
   24 WRITE(5,101)(Y(I),(X(I,J),J=1,ICOL))
      DO 10 I=1,ICOL
      XTY(I)=0.
      DO 10 J=1,ICOL
   10 XTX(I,J)=0.
      DO 11 I=1,ICOL
      DO 11 J=1,IROW
   11 XTY(I)=XTY(I)+Y(J)*X(J,I)
      DO 15 I=1,ICOL
      DO 15 J=1,ICOL
      DO 12 K=1,IROW
   12 XTX(I,J)=XTX(I,J)+X(K,I)*X(K,J)
   15 STO(I,J)=XTX(I,J)
      WRITE(5,95)
      DO 25 I=1,ICOL
   25 WRITE(5,102)(XTY(I),(XTX(I,J),J=1,ICOL))
      Z(1,1)=1.
      DO 21 J=2,ICOLX
      Z(J,1)=XTY(J-1)
   21 Z(1,J)=XTY(J-1)
      DO 22 I=2,ICOLX
      DO 22 J=2,ICOLX
   22 Z(I,J)=XTX(I-1,J-1)
C
C     RIDGE SUBROUTINE
C
      DO 66 L=1,51
      AZ=0.01*(L-1)
      DO 16 I=1,ICOL
      DO 17 J=1,ICOL
   17 XTX(I,J)=STO(I,J)
   16 XTX(I,I)=XTX(I,I)+AZ
      V(1)=3
      CALL GJR(XTX,MCOL,MROW,ICOL,ICOL,$97,JC,V)
      DO 13 I=1,ICOL
      BETA(I)=0.
      DO 13 J=1,ICOL
   13 BETA(I)=BETA(I)+XTY(J)*XTX(I,J)
      BETAO=0.
      DO 14 I=1,ICOL
      BETA(I)=BETA(I)*SUMSQY/SUMSQX(I)
   14 BETAO=BETAO+BETA(I)*M(I)
      BETAO=YBAR-BETAO
      IF(AZ.EQ.0)WRITE(5,108)
      IF(AZ.EQ.0)WRITE(5,110)
      C=EXP(V(2))
      WRITE(5,98)AZ
```

```
      WRITE(5,102)BETAO,(BETA(I),I=1,ICOL)
      WRITE(5,104)C
      R2=BETAO*T
      DO 44 I=1,ICOL
   44 R2=R2+RSTO(I)*BETA(I)
      R2=(R2-T*T/ROWS)/ISS
      R2A=1.-((ROWS-1.)/(ROWS-COLX))*(1.-R2)
      WRITE(5,109)R2A,ICOLX
      DO 2 I=1,IROW
      PR(I)=0.
      DO 1 J=1,ICOL
    1 PR(I)=BETA(J)*XPR(I,J)+PR(I)
      PR(I)=PR(I)+BETAO
    2 ER(I)=AC(I)-PR(I)
      DW1=0.
      DW2=0.
      DO 3 I=2,IROW
    3 DW1=DW1+(ER(I)-ER(I-1))**2
      DO 4 I=1,IROW
    4 DW2=DW2+(ER(I)*ER(I))
      DW=DW1/DW2
      WRITE(5,112)DW
   66 CONTINUE
C
C               EIGVECTOR     SUBROUTINE
C
      CALL TRIDMX(ICOLX,MCOL,Z,D,B)
      CALL EIGVAL(ICOLX,EIGNVL,D,B,T1,T2)
      WRITE(5,92)
      WRITE(5,102)(EIGNVL(I),I=1,ICOLX)
      CALL EIGVEC(ICOLX,MCOL,Z,D,B,EIGNVL,EIGNVC,T1,T2)
      WRITE(5,94)
      DO 26 I=1,ICOLX
   26 WRITE(5,102)(EIGNVC(I,J),J=1,ICOLX)
      WRITE(5,111)
      DO 85 K=1,ICOLX
      IF(K.EQ.1)GOTO 89
      ITEST=ICOLX-K+2
      IF(ABS(EIGNVL(ITEST)).GT.TLAM)STOP
      IF(ABS(EIGNVC(1,ITEST)).GT.TGAM)GOTO 85
      IF(K.EQ.2)WRITE(5,105)
      WRITE(5,106)ITEST
   89 DO 71 NDX=1,NIT
      AK=RIT*(NDX-1)
      WRITE(5,98)AK
      IF(K.GT.1)GOTO 77
      S1(NDX)=0.
      DO 82 J=1,ICOLX
   82 S1(NDX)=S1(NDX)+EIGNVC(1,J)**2/(EIGNVL(J)+AK)
      DO 74 I=1,ICOL
      S2(I,NDX)=0
      DO 83 J=1,ICOLX
   83 S2(I,NDX)=S2(I,NDX)+EIGNVC(1,J)*EIGNVC(I+1,J)/(EIGNVL(J)+AK)
   74 BETAE(I)=-S2(I,NDX)/S1(NDX)
      GOTO 87
   77 S1(NDX)=S1(NDX)-EIGNVC(1,ITEST)**2/(EIGNVL(ITEST)+AK)
      DO 86 L=1,ICOL
      S2(L,NDX)=S2(L,NDX)-EIGNVC(1,ITEST)*EIGNVC(L+1,ITEST)/(EIGNVL(ITEST)+AK)
     UT)+AK)
   86 BETAE(L)=-S2(L,NDX)/S1(NDX)
   87 BETAEO=0
      DO 84 I=1,ICOL
      BETAE(I)=BETAE(I)*SUMSQY/SUMSQX(I)
   84 BETAEO=BETAEO+BETAE(I)*M(I)
      BETAEO=YBAR-BETAEO
   81 WRITE(5,102)BETAEO,(BETAE(I),I=1,ICOL)
      R2=BETAEO*T
      DO 50 I=1,ICOL
   50 R2=R2+RSTO(I)*BETAE(I)
      R2=(R2-T*T/ROWS)/ISS
      R2A=1.-((ROWS-1.)/(ROWS-COLX))*(1.-R2)
      WRITE(5,109)R2A,ICOLX
   71 CONTINUE
   85 CONTINUE
   97 WRITE(5,100)
   92 FORMAT(1H1,6X,'EIGEN-VALUES IN DESCENDING ORDER',/)
   93 FORMAT(////,20X,'STANDARDIZED DATA MATRIX',/)
   94 FORMAT(////,5X,'EIGEN-VECTORS RELATED TO ABOVE EIGEN-VALUES(PRINTE
     1D IN COLUMNS)',/)
   91 FORMAT(1H1,20X,'ORIGINAL DATA MATRIX',/)
   95 FORMAT(/,5X,'CORRELATION COLUMN BETWEEN X AND Y AND CORRELATION MA
     1TRIX BETWEEN X AND X',/)
   98 FORMAT(////,5X,'VALUE OF BETA COEFFICIENTS WHEN K EQUALS',1X,F9.2
     1,/)
  100 FORMAT(////,20X,'XTX SINGULAR')
  101 FORMAT(1X,10(F10.2))
  102 FORMAT(1X,10(F13.5))
  103 FORMAT(1X,13(F10.4))
  104 FORMAT(/,4X,'VALUE OF MODULUS OF DETERMINANT =',F12.8,1X)
  105 FORMAT(1H1,////,5X,'CRITERION-EXCLUDE LAMDA(I) IF LAMDA(I).LT.0.0
     15 AND GAMMA(0,I).LT.0.1')
  106 FORMAT(//,2X,'EXCLUDE LAMBDA',I3)
  108 FORMAT(1H1,///,5X,'BETA COEFFICIENTS WRITTEN OUT FROM BETA(0) TO B
     1ETA(P)',/,5X,52('*'))
  109 FORMAT(/,4X,'VALUE OF R-SQUARED =',F7.5,1X,'(ADJUSTED FOR ',I2,1X,
     1'DEGREES OF FREEDOM)')
  110 FORMAT(1H1,///,5X,'RIDGE ESTIMATES OF BETA-COEFFICIENTS',/,5X,36('*'))
  111 FORMAT(1H1,///,5X,'CHARACTERISTIC-ROOT ESTIMATES OF BETA COEFFICIE
     1NTS',/,5X,50('*'))
  112 FORMAT(/,4X,'           D.W. =',F7.5)
      END
```

```
@RUN BARWIZ,A0227-RG15/CCON1,WYMER,S20,20

@ASG,A  BARR*WIZ.

@DATA,IL BARR*WIZ.
DATA T7 RL70-5 09/14-11:11:28
        @ASG,A BARR*DATAFIL66.
        @ASG,A BARR*T2.
        @ASG,A TRANSF.
        @USE 11,BARR*T2.
        @XQT TRANSF,ABSOLUTE
        START,3                    MACRO-MODEL
              1    35    16    60    60     1     0     0     1     1
           31   32   33   34   35   36   37   38   39   40   41   42   43   44   45   46   47   48   49   50
           51   52   53   54   55   56   57   58   59   60   61   62   63   64   65
        (5(10(F8.2),/),10(F8.2))
        MODEL MACRO
        @ADD,P  BARR*DATAFILE6.
        ENDA
        X60=X60/100.
        X32=(X32+X33)/X60
        X38=(X38-X62)/X60
        X40=X40/X60
        X44=(X44+X45-X48)/X60
        X41=X41/X60
        X35=X35/X60
        X1=X32
        X2=X38
        X3=X63
        X4=X40
        X5=X44
        X6=X65
        X13=X41
        X14=X35
        DSEAS     1     6     4
        DSEAS    13    14     4
        COSMA     1     6     1
        COSMA    13    14     1
        LAG       1     7
        LAG       2     8
        LAG       3     9
        LAG       4    10
        LAG       5    11
        LAG       6    12
        LAG      13    15
        LAG      14    16
        REDUC     2     1    58
        FINAL
        FINISH
END DATA.

@FIN
```

```
@ASG,A   BARR*WIZ.

@DATA,IL  BARR*WIZ.
DATA T7 RL70-S 09/14-11;08:24
        @ASG,A BARR*T2.
        @USE  11,T2.
        @USE  11,BARR*T2.
        @ASG,A  WYMER*RESIMUL.
        @XQT  WYMER.RESIMUL.ABSOLUTE
        START,1                           MODEL-MACRO
           1    6    7   59   12   18    2    1    1    1    3    1
           4    2
        F    1   1P    1   7F   -2  13F   -3   2F   -4   8F    4    5
        F    6   3P    3   9F   -7   2F   -8    8
        F   10   2F  -11   1F  -12   3F  -12   7P    5   8F  -12  12F   14   6F  -12  -11
        F  -16  13P    6  10P    5    4
        C    1   4F  -17   1P   -8   3P   -8   7P   -8  12P   -8   11
        C    1   5F  -18   2P   -9   8P   12   13
        C    1   6C   -2   2C   -1   8C   -1   4C    1   3C    1  12C    1  11C    2   -1
        C    1    7
        F1=C1+C2P1$F2=P1P10$F3=P1P2$F4=C1P1P2$F5=P1P12$F6=C1+C2P3$F7=C2P3P4$F8=P3P4$F9=C
        1P3$F10=C1+C2P5$F11=C2P5+C2P6P7$F12=P5+P6P7$F13=P5+C2P6P7$F14=C2P6$F15=C1P12$F16
        =C1P11$F17=C2P8$F18=C2P9$S
        .7253       .7011       .1128       11.69       1.          1.          1.          .218
        .149        82.89       59.18       103.411
        1.          .5
           1   16   59    1    0    0    1
           1    2    3    4    5    6   21   22   23   24   25   26   27   28   29   30
        MACRO-MODEL
        X1=X1-X21
        X2=X2-X22
        X3=X3-X23
        X4=.5(X4+X24)
        X5=.5(X5+X25)
        X6=X6-X26
        X25=.5(X27+X29)
        X26=.5(X28+X30)
        X27=.785
        FINAL
           0    0    0    0
        FINISH
END DATA.
```

```
@RUN BARWIZ,A0227-R015/ECONI,WYMER,S40,20

@ASG,A  BARR*WIZ.

@DATA,IL BARR*WIZ.
DATA T7 RL70-5 09/14-11:08:06
          @ASG,A BARR.DATAFILE6.
          @ASG,A BARR.T3.
          @USE 11,BARR.T3.
          @XQT TRANSF.ABSOLUTE
          START,3                  MODEL  BARWIN
               1   35   27   60   60    1    0    0    1    1
              31   32   33   34   35   36   37   38   39   40   41   42   43   44   45   46   47   48   49   50
              51   52   53   54   55   56   57   58   59   60   61   62   63   64   65
          (5(10(F8.2),/),10(F8.2))
          MODEL  BARWIN
          @ADD,P BARR.DATAFILE6.
          ENDA
          PRINT   31   65
          X7=X60/100.
          X1=LOG((X32+X33)/X7)
          X2=LOG(X38/X7)
          X3=LOG((X38-X44-X45+X55)/X7)
          X4=LOG(X40/X7)
          X5=LOG(X41/X7)
          X6=LOG(X65)
          X8=LOG(X61/X7)
          X9=LOG((X32+X33+X61+X35+X41)/X7)
          X10=LOG(X50/X7)
          X11=LOG(X42/100.)
          X12=LOG(X43/100.)
          X7=LOG(X7)
          X26=X51/400.
          TREND   13
          DSEAS    1    6    4
          DSEAS    8   10    4
          COSMA    1   12    1
          COSMA   26   26    1
          LAG      1   14
          LAG      2   15
          LAG      3   16
          LAG      4   17
          LAG      5   18
          LAG      6   19
          LAG      7   20
          LAG      8   21
          LAG      9   22
          LAG     10   23
          LAG     11   24
          LAG     12   25
          LAG     26   27
          REDUC    5    1   56
          FINAL
          FINISH
END DATA.

@FIN
```

```
@ASG,A BARR*WIZ.
@DATA,IL BARR*WIZ.
DATA T7 RL70-5 09/14-11:12:00
        @ASG,A BARR*T3.
        @USE 11,BARR*T3.
        @ASG,A CONTINEST.
        @ASG,A BARR*C1.
        @USE 15,BARR*C1.
        @ASG,A WYMER*RESIMUL.
        @XQT WYMER*RESIMUL.ABS
        START,1                 CONTINEST
            1    5   15   56   15   17    4    1    0    1    3    1
            5
        F   1   1P   1   6F  -2  11F  -3   19
        F   4   2P   2   7F  -5  10F   6  17F  -6  18F  -7   19
        F   8   3F  -9  13F -10  16F  10  14P   3   8F -11   19
        F  12   4F -13  15F  13  14P  -4  12P   4   9F -14   19
        F  15   5P   5  10F  16  15F -16  14P  -5  12F -17   19
        F1=C1+C2P1$F2=P1P6$F3=P1P12$F4=C1+C2P2$F5=P2P7$F6=P2P8$F7=P2P13$F8=C1+C2P3$
        F9=P3P11$F10=P3P9$F11=P3P14$F12=C1+C2P4$F13=P4P10$F14=P4P15$F15=C1+C2P5$
        F16=C3P5P10$F17=C3P5P15+C4P5$$
        .31046    .1657     2.6443    .4081     .59028    .95459    1.74836   30.
        .69916    1.        .00969    -.04027   -6.9729   6.0611    -1.6
        1.        .5        .22       -.58
          1  27  56   1   0   0   1
          21  22  23  24  25  26  27  28  29  30  31  32  33  34  35  36  37  38  39  40
          41  42  43  44  48  46  47
         MODEL  BARJIN
        X1=X21-X34
        X2=X28-X41
        X3=X25-X38
        X4=X24-X37
        X5=X22-X35
        X26=.5(X23+X36)
        X48=X29
        X29=.5(X27+X40)
        X21=X34
        X49=X33
        X33=X27-X40
        X22=X41
        X23=X38
        X24=X37
        X25=X35
        X27=.5(X48+X42)
        X28=X49
        X30=.5(X31+X44)
        X31=.5(X32+X45)
        X32=.5(X46+X47)
        X34=.785
        FINAL
            0    0    1    5
        FINISH
        @XQT CONTINEST.ABS
        START,1
            5    0    2    0    1    0    1
END DATA.
```

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